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THE SYRACUSE UNIVERSITY "MADISON PROJECT"

BY

ROBERT B. DAVIS

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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ROBERT B. DAVIS, Syracuse University

This is an extremely brief report on the Syracuse University "Madison Project." Longer accounts appear in the M.I.T. *Technology Review* and elsewhere (see references).

The project is concerned with mathematics teaching, particularly in grades 3–9 (but also, to a lesser extent, with the process of learning mathematics on the college and graduate levels). It operates mainly in New York State and in Connecticut. (The name, incidentally, derives from the name of the first school in which the project operated, Madison Junior High School in Syracuse, New York.)

The main objective of the project is a behavioral study of how people lea[illegible] mathematics. A two-year program in algebra, suitable for presentation to chi[illegible]dren beginning as early as grade 3, is one concrete result of the project's studies. A workbook to accompany this algebra course is another concrete by-product of project research.

The workbook. The workbook provides for a two-year sequence in algebra, with the following features:

(i) An unusual effort is made to present basic mathematical concepts clearly, and to keep the logic as nearly 100% honest as possible. This latter requirement implies that most of the logic is inductive, by generalization from experience, but a small amount of very simple deductive logic is included (*e.g.*, one of the rare pieces of deductive logic—if a single substitution into an open sentence results in a false statement, the open sentence was not an *identity*, because for identities *every* substitution produces a true statement).

(ii) The language itself is kept casual and unobtrusive. Mathematical concepts are developed from experience with mathematical situations, not by exposition.

(iii) The student learns mainly from his experiences in analyzing a sequence of problems. For example, by plotting linear functions by trial and error, he discovers how to identify and make use of the slope coefficient. We know from experience that even low-I.Q. children can readily discover these things for themselves, if we merely give them an appropriate sequence of problems. *The course includes extremely little exposition*—the workbook itself is mainly a sequence of questions, such as: (1) Graph $y=2x+1$; (2) Graph $y=3x+1$; (3) Graph $y=5x+4$; etc.

(iv) This course has been taught with notable success to gifted fourth and fifth graders, to low-I.Q. 7th graders, and to many other groups, beginning as early as grade 3, and including a full range of I.Q.'s from low-I.Q. classes to gifted classes.

(v) The workbook deals only with algebra. It can be used as occasional enrichment material: a fourth-grade teacher, for example, might introduce some one or two topics from the workbook. In this way, both teachers and school systems can introduce more algebra into the grades by a gradual process. This flexibility is a big advantage over a closely integrated program which might have to be adopted as a whole or else not at all.

(vi) The program lays great stress on identifying the basic ideas (and especially the *hard* ideas) of mathematics and introducing them *early*, so that they may be progressively refined in many subsequent years of study. The project advocates a first appearance of the concept of limit, for example, as early as grade 4, in simple form. Specifically, we can seek numbers which, when multiplied by themselves, result in a product of 2. The germ of the entire concept of limit is contained in this problem, and can be brought out in subsequent study during the next six or eight years.

(vii) Since mathematics, we believe, consists in large measure of mastery of the art of creative problem solving, students practice this at every step of their study. This is a corollary of our nonexpositional approach of teaching by asking a sequence of questions.

Why algebra in the grades? We have many reasons for wishing to begin the study of algebra as early as grade 3 or 4. To list just a few:

(i) We believe a great deal of algebra is *easier* than a great deal of arithmetic.

(ii) We believe that many algebraic concepts and tools lie at the heart of arithmetic, and can be used both to give deeper insight into arithmetic and to simplify the study of arithmetic.

(iii) We believe that continual practice in creative problem solving is essential to the true nature of mathematics, and that algebra provides more (and easier) material for the practice of this art than arithmetic does.

Other aspects of the project. As mentioned above, the workbook is merely a concrete by-product of project research. We believe that problems in the teaching of mathematics should not be regarded as national crises to be overcome by gigantic crash programs. Rather, we insist that a wiser approach is to regard mathematics education as a uniquely fruitful field for studying the behavior of the human mind; curricula, courses, tests, and so on should appear as "applied" bonuses that stem from a program of basic research.

The experience of four years has convinced us that in working with the schools it is necessary to operate on a regional basis, with collaboration between teachers and university mathematicians. A program brought in from outside cannot and will not be installed successfully within a local school. If you want a better program in a school, you have to cultivate it and grow it right there on the spot.

The use of films. Many observers have been impressed by the project's success in getting fourth graders *to discover for themselves* the coefficient rules for quadratic equations, and so on. We feel this success is closely related to the conduct of our classes. This creates two problems: First, how can we show teachers just what we mean when we speak of "discovery method," "making mathematics an intellectual challenge rather than dreary routine," and so on? Second, how can we study project classes in order to determine why they *do* work so well? We hope that films (with sound) may provide an answer to both questions.

References

1. James Canan, Junior Highers are Introduced to New Methods, Binghamton Sunday Press, September 27, 1959, p. 3-C.

2. Gene Currivan, New Arithmetic Methods Urged to Stimulate Interest at Early Age, New York Times, July 26, 1959 (education page). ~~323.08~~

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3. Robert B. Davis, The Mathematics Revolution: Causes and Directions (Ch. VII of Frontiers of Secondary Education III, Paul M. Halverson, Ed.), Syracuse University Press, 1958.

4. ———, Seventh-grade algebra: The Madison project, New York State Mathematics Teachers Journal, vol. 9, 1959, pp. 5–8.

5. ———, How do we learn mathematics? M.I.T. Technology Review, vol. 62, 1959, pp. 28–30.

6. Cynthia Parsons, Algebra as Presented to Fourth Graders is Grasped with Enthusiasm, Christian Science Monitor, January 9, 1960, p. 7.

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Syracuse University

MADISON PROJECT

Workbook
1st Course

Teacher's Manual

AVAILABLE FROM: Madison Project
Mathematics Department
Syracuse University
Syracuse 10, New York

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There are two facts about teachers:

1) They are busy every minute.
2) They must be allowed to teach the way that suits them best.

We have tried to keep these facts in mind in preparing this teacher's manual for the Madison Project Workbook.

We will try to make suggestions clearly and briefly. (Unfortunately, this makes them sound dogmatic.)

But if a teacher prefers to ignore certain of our suggestions, and use her own methods instead, she is certainly entitled to do so. That is her right.

Before using the Madison Project Workbook in class, please read this section:

1) The absence of exposition. In our use of the workbook, we almost never tell the students what to do, nor how to do it. Instead, we ask questions.

The student learns by thinking through the questions himself(we call this the "Discovery Method"), or, in some cases, by imitating the behavior of the teacher or of other students.

2) Early introduction of concepts . We try to get the student thinking about the basic concepts as early as possible. We avoid the use of extensive vocabulary. Names, and also calculations, tend to obscure the concepts, and it is the concepts that we want the student to think about. He can learn names and calculations later -- after he has thought about the concepts in a creative way for some time. (By then he is ready for names and calculations, and they won't hurt him.)

3) Conversations, not lectures. We usually conduct our classes with a maximum of student participation. If a student makes an error, we usually wait until some other student notices the error and corrects it. Every effort is made to get the students thinking and talking. We do not want them listening and accepting if we can avoid it.

4) Success experiences. In our experience, nearly every student answer has some merit--if only as a courageous try. We respond to every student answer as we believe scientists should--it's an answer, and it deserves respect. If it turns out to be right, it helps us with our work. But, even if it turns out to be wrong, it usually adds to our understanding--and so it, too, helps us along with our task.

We try to avoid moral judgments, or words which suggest moral judgments. We prefer to say "Yes, that works")if it turns out to), rather than saying "Good" or "Right"; if an answer turns out not to work, we usually say something like (perhaps) this:

"Well, that doesn't work; do you have any other suggestions?"

We want the students to feel we are partners in an experience of intellectual discovery; we do not want them to feel that we are standing over them waiting to pass judgment on them.

5) The light touch. The Madison Project introduces mathematical topics years earlier than most curricula. This gives us, among others, the advantage of being under absolutely no pressure. It is not necessary to be tedious, or comprehensive.

We explore the fascinating concepts of mathematics as long as they fascinate us; when they threaten to become routine, we turn to a new topic.

After all, our real purpose is to get students thinking, on their own, about the basic concepts of mathematics. We want them to enjoy it. We prefer to end a lesson while the students are still begging for more.

LESSON I - CHAPTER I

Chapter I is the first lesson. We cover this easily in 45 or 50 minutes. (This is a good test of the "light touch." We try not to exhaust the subject, the students, or ourselves.)

The object of this lesson is to start students thinking about the concepts of equation, open sentence, truth set, and inequality.

It is not necessary to work every example. The workbook probably contains more problems than you will actually want to use in this lesson.

It is not necessary to use the words "equation" or "inequality", although you may prefer to. "Open sentence" includes both of these.

In any event, we suggest that it is best to skip lightly over nomenclature. The names of things are not really very important, and it is best not to waste time on them.

1. Equations, Open Sentences, and Inequalities

Can you fill in the proper number in the squares?

1) $3 + \square = 5$

Ans. 2

2) $2 + \square = 6$

Ans. 4

3) $5 + \square = 11$

Ans. 6

4) $3 + \square = 16$

Ans. 13

5) $8 + \square = 21$

Ans. 13

Students usually find these easy, and can answer them immediately.

6) $3 + (2 \times \square) = 7$

Ans. 2

We suggest: do not explain the use of parentheses. If a student suggests '5' as the answer, we simply follow the suggestion:

$$3 + (2 \times \square) = 7$$

We might say: " 2 x 5 is ten.

Then we have to add three:

$$3 + 10 \neq 7$$

So this doesn't work; do you have any other suggestions?"

The way we substitute "5" on the board is important. Although we do this with a casual air, we actually realize the student is going to imitate our use of parentheses.

(Remember, we don't tell him about the use of parentheses--he has to observe for himself how we use them.)

A step-by-step description of what we might write and say, if a student suggested substituting "5":

We write

First, we already have the equation on the board

$$3 + (2 \times \square) = 7$$

We say

We substitute as requested:

"All right, let's try 5 . . ."

$$3 + (2 \times \boxed{5}) = 7$$

"Two times five is ten . . ."

$$3 + (\underset{10}{2 \times \boxed{5}}) = 7$$

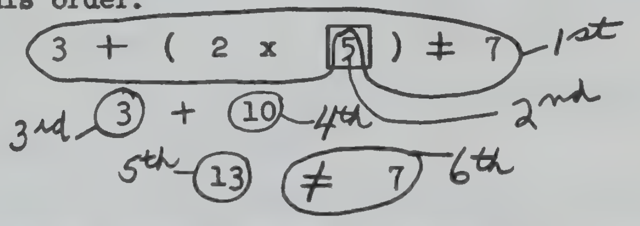
We say	**We write**
". . . and we have to add 3 . . ."	$3+(2\times\boxed{5})=7$
". . . 3 and 10 are thirteen . . ."	$3+10$
	$3+(2\times\boxed{5})=7$
n	$3+10$
	13
". . . which is not equal to 7 . . ."	$3+(2\times\boxed{5})=7$
	$3+10$
	$13\neq 7$

" . . . so 5 doesn't work; do you have any other suggestions? . . ."

What we write at each step has been indicated on the right. In other words, we write:

$$\begin{gathered}3+(2\times\boxed{5})=7\\3+10\\13\neq 7\end{gathered}$$

which we write in this order:



This is actually a natural order; you will probably find yourself using it even if you don't try.

Of course, the right answer to number 6 is:

$$3+(2\times\boxed{2})=7$$

$$\begin{gathered}3+(2\times\boxed{2})=7\\4\end{gathered}$$

$$\begin{gathered}3+(2\times\boxed{2})=7\\3+4\end{gathered}$$

$$\begin{gathered}3+(2\times\boxed{2})=7\\3+4\\7=7\end{gathered}$$

7) $4+(2\times\square)=10$ Ans. 3

8) $3+(3\times\square)=18$ Ans. 5

9) $2+(3\times\square)=14$ Ans. 4

10) $1+(2\times\square)=6$ Ans. 2½

11) $1+(2\times\square)=4$ Ans. 1½

12) $2+(2\times\square)=11$ Ans. 4½

Students usually find these easy, once they have seen us work with number 6.

Students sometimes say that "there is no answer to this problem." This a clever

Sentences or statements may be true, false, or open.

"Washington was once President of the United States." This is a true statement.

remark--the student who says this has discovered for himself the fact that there is no whole number solution. Ahh. . . but how about fractions? Suggestion: When, as with problems 10, 11, and 12, our first few guesses are not likely to work out, it is well to suggest that students keep two lists: one labelled "too small", and the other labelled "too large:"

$$1 + (2 \times \square) = 6$$

Too small	Too large
2	3
1	4

We shall make more use of these "too small" and "too large" tables in later lessons.

Sentences--true, false, and open. Students usually find this easy.

(bottom of page 2, and pages 3, 4, and top of page 5)

(bottom of page 5--problems 19, 20, and 21) Whatever the class says about these is usually acceptable. We intend the light t~~w~~ouch here, and don't mean to insist upon the facts.

While we wouldn't fight over it, it is our opinion that:

19 should be called true
20 should be called false (Marx was a German.)
21 should be called true

(pages 6,7) It is common to use brackets in writing sets. Thus the set of even numbers could be written: $\{2, 4, 6, 8, \ldots\}$
(The final three dots indicate an infinite set, which we shall discuss later.)

Similarly, the set of integers from 1 to 10 inclusive could be written:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

A set is given by a list -- the order is not important. We could write the last set above as: $\{2, 7, 3, 4, 1, 9, 8, 5, 6, 10\}$

Students often dictate sets in such an order. This is perfectly all right -- it may even indicate more thought than if they merely rattled them off in order.

(pages 7, 8) Examples 28 and 29. Write answers in set notation, using brackets:

28)	{Andy}
29)	{Miss Brooks}

Example 30: Here the truth set probably has several elements:

$\{$Mary, Joe, Marie, Pat, John$\}$

If any one of these is substituted, the sentence becomes true:

for example: Marie sits in the front row.

(page 9)

35) Ans. $\{0, 1, 2\}$

36) Ans. $\{2\}$

37) Ans. $\{2, 3, 4, 5,\}$

38) Ans. $\{9\}$

39) Ans. $\{3, 4, 5, 6, 7\}$

LESSON II -- CHAPTER II

This lesson has four objectives:

1) Review of truth sets from previous lesson (problems 1-6).

2) Introduction of open sentences with two placeholders (also called "equation in two variables.")

3) Introduction of tables and graphs.

4) Giving thr brightest students an opportunity to discover a pattern in graphs (which in fact is actually the concept of slope). We shall explain this further in a moment.

(page 10)

1) Ans. $\{5\}$
2) Ans. $\{4\}$
3) Ans. $\{1\}$
4) Ans. $\{7\}$
5) Ans. $\{3\frac{1}{2}\}$
6) Ans. $\{1\ 1/3\}$

We suggest that on every problem of this type if our first guess of guesses do not work out, it is extremely advisable to make two lists, one headed "too small: and the other "too large."

Sometimes . . . etc.

For example, if, on problem 1), a student suggested that we try "6", we would find that

$$1+(3\times\square)=16$$
$$1+18$$
$$19\neq 16$$

and it is clear that "6" is too large.

So we start our lists:

Too small	Too large
	6

This means that we do not need to try any numbers larger than 6. It makes our work much more efficient.

(Of course, the only method we have for solving equations at this stage is trial and error. The use of "too small" and :"too large" lests permits us to use trial and error in a systematic way.

In problems 5 and 6, these lists are especially helpful. Let us work out 6) as an example of how the lists can help:

6) $1+(3\times\square)=5$

Suppose a student suggests that we try "3". We carry out his suggestion:

$$1+(3\times\boxed{3})=5$$
$$1+9$$
$$10\neq 5$$

We see that 3 is too large.

We start our lists.

Too small	Too large
	3

Now-suppose a student suggests "4". We could ask the class if they believe we need to try "4" -- (a "poker" voice that doesn't give away the answer is a big asset here).

Probably someone in the class will realize that if 3 is too large, then 4 must also be too large.

Suppose a student now suggests 2. We substitute:

$$1+(3\times\boxed{2})=5$$
$$1+6$$
$$7\neq 5$$

. . .and we find that "2" is too large. We put it on our list:

Too small	Too large
	3
	2

Probably the class will now suggest that we try "1". We do:

$$1+(3\times\boxed{1})=5$$
$$1+3$$
$$4\neq 5$$

. . .and we find that "1" is too small!

Adding it to our lists:

Too small	Too large
1	3
	2

The class will probably suggest that we try $1\frac{1}{2}$. All right, let's try $1\frac{1}{2}$:

$$1+(3\times\boxed{1\tfrac{1}{2}})=5$$
$$1+4\tfrac{1}{2}$$
$$5\tfrac{1}{2}\neq 5$$

. . .and $1\frac{1}{2}$ is too large:

So our list grows:

Too small	Too large
1	3
	2
	$1\frac{1}{2}$

Now we know that we must try numbers bigger than 1, but smaller than $1\frac{1}{2}$.

This is a test of the students' ability to compare the size of different fractions:

Of course, the right answer is 1 1/3. Substituting:

$$1 + (3 \times \boxed{1\tfrac{1}{3}}) = 5$$

$$1 + 4$$

$$5 = 5$$

So this answer works out!

(We suggest that you do not tell students the right answer. Wait until thay guess a number that actually works out when you substitute.)

(page 11)

7) Ans. Yes, because $\overset{5}{\triangle} = \boxed{2} + 3$ is a true statement.

You may wish to use some examples in words, such as:

"___________" is a friend of(___________).

The table for the truth set might then look something like this:

"___________"	(___________)
Mary	Joan
Joan	Mary
Harold	Gene
John	Albert
Gene	Harold
Albert	John
	and so on...

(page 12)

8)

□	△
1	4
2	5
3	6
4	7
5	8
6	9
7	10
8	11
½	3½
1½	4½
3½	6½

(page 13)

9)

□	△
1	6
2	7
3	8
4	9
10	15
15	20
1½	6½
18½	23½

(page 14)

10)

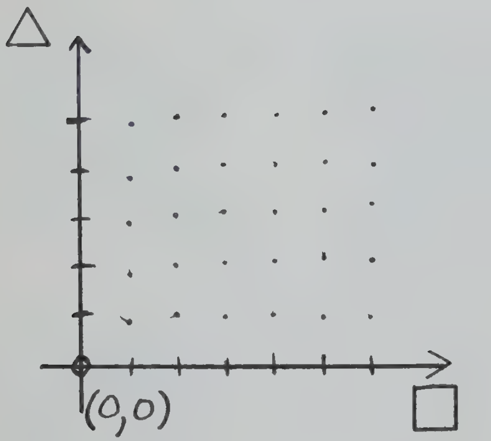
□	△
1	3
2	5
3	7
4	9
6	13
7	15
8	17
½	2
2½	6
3½	8
4½	10

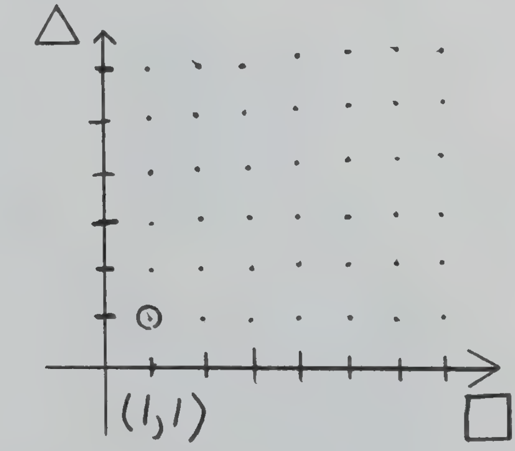
The exposition of graphing in the Workbook is deliberately brief. This is perhaps a place where clear, brief exposition by the teacher might be in order.

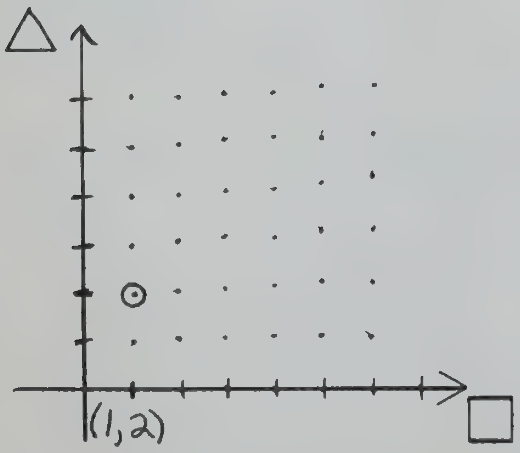
(Notice that this is virtually the first place in the workbook where the teacher should tell the class how to do something..)

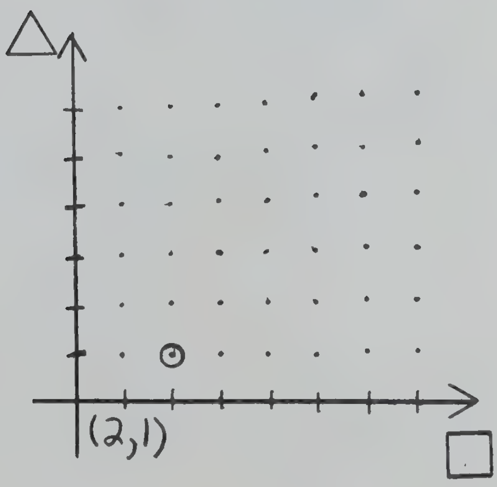
(pages 15-16)

You may wish to look at a few more correctly graphed points:









Probably the most common errors in graphing points are these two:

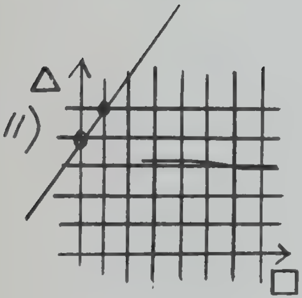
1) reversing the coordinates (for example, putting (1, 2) where (2, 1) should be);

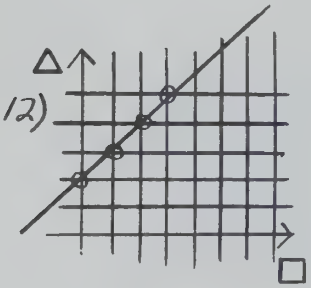
2) beginning the counting wrong, so that (3, 4) might be erroneously located where (4, 4) should be, or where (2, 4) should be. (Or, perhaps where (4, 5), (3, 5), (2, 3), or (3, 3) should be.)

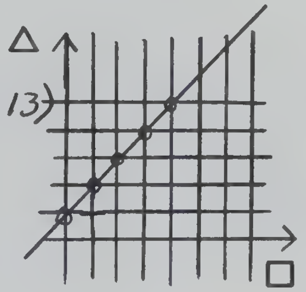
(page 16, bottom of 17)

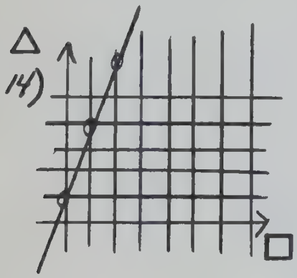
There are two opportunities on the page.

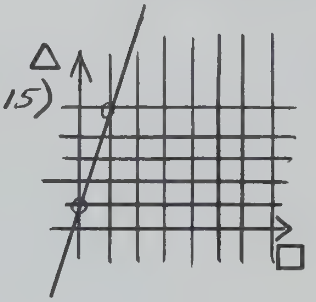
1) We hope most students will learn to locate points correctly in graphing these truth sets. Corect graphs look like this:

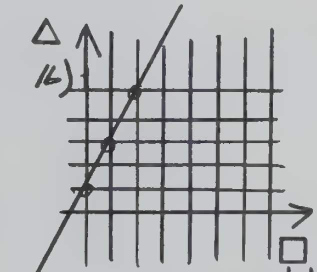


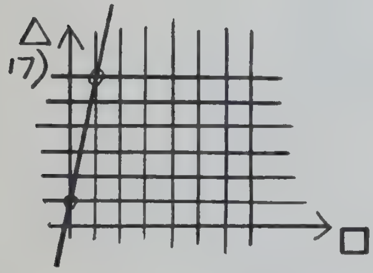


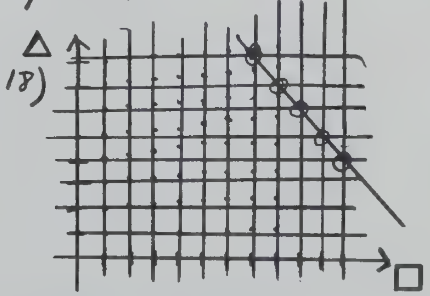








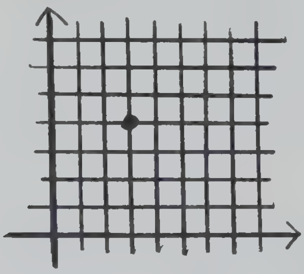




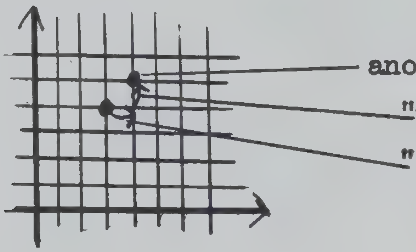
2) There is, however, a further opportunity: we hope that some bright students will discover the pattern for each problem.

There are two possible discoveries:

i) Each problem has some pattern: for example, in problem 13, we can get from one point to another by going "over one(to the right) and up one." Thus, (3, 4) is an element of the truth set.

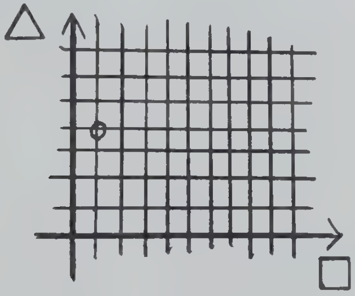


We can find another point by going "over one and up one."

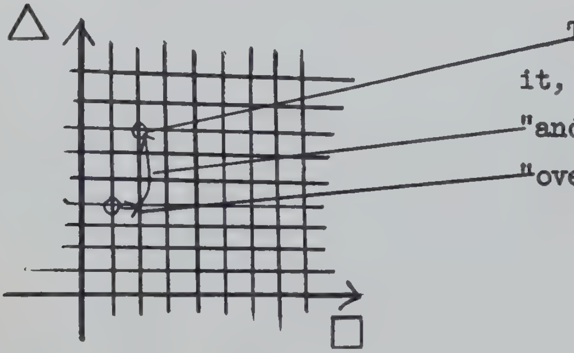

another point is here.
"up one"
"over one" (to the right)

In problem 14, the pattern is "over one and up three";

Starting at (1, 4), which is an element of the truth set,



We go over one, and up three:



The new point is here (5, 7); it, also, satisfies the equation.
"and up three"
"over one" (to the right)

Some bright students may discover these patterns as they go about their task of locating points by trial and error.

ii) The second possible discovery is this: The pattern on the graph is related to one of the numbers in the equation:

For $\triangle = (\underline{\underline{3}} \times \square) + 1$, the pattern is "over one, up *three*."

For $\triangle = (\underline{\underline{2}} \times \square) + 1$, the pattern is "over one, up *two*."

These are the first major discoveries of the course, so we'd like to repeat the basic ideas:

1) All that we *tell* the students is how to locate a point on the graph. If students do *not* discover the patterns, we do *not* tell them about the patterns.

2) Usually, as they work along plotting points by trial-and-error, one or two bright students will discover the pattern in the problem they are working.

3) If they do, we let the whole class know that "Jane" (or whoever) has discovered a pattern in problem 13. We might not immediately tell the class what the pattern is, until several other students have independently discovered "the pattern in problem 13."

4) When a number of students have "discovered the pattern in problem 13" *independently,* we might let the rest of the class in on the secret, by showing them *exactly what the pattern is*.

5) Then we might ask them to "see if Jane's pattern also works in problem 14." If not (and of course it doesn't), we ask them: "Can you modify Jane's pattern so that it *will* work in problem 14?"

6) Once a student is sure of the pattern in a problem, he can use it to save time in plotting further points.

7) Students do *not* usually discover the relation between the pattern and the number

$$\triangle = (\underline{\underline{3}} \times \square) + 1$$

in the equation. Well, that's life. WE DO NOT TELL THEM!

This section is one the real opportunities for the teacher to practice "the light touch" and to use the "discovery method." Good luck!!

(You may notice that in problem 18, the pattern is:

"over one and *down* one.")

LESSON III - CHAPTER III

We suggest betinning this lesson with a very brief review of open sentences, truth sets, and graphs.

The main point of this third lesson is the Point-Set Game. We might now discuss two matters:

i) How we play this game in class.

ii) Why we regard games such as this as important parts of our work.

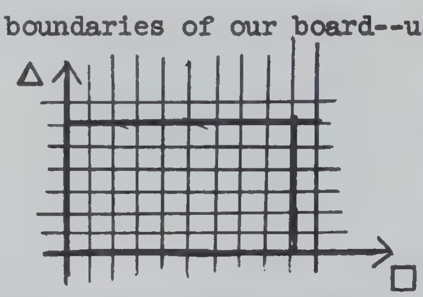
First: How We Play.

1) We choose two team captains.

2) The captains pick up sides.

3) One team will use the symbol ● to mark its points, and the other team will use the symbol ○.

4) We mark off the boundaries of our board--usually something like this:

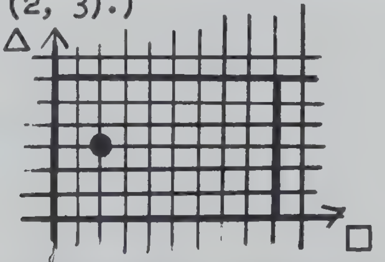


(At this stage we are limited to non-negative integers.)

5) The first player on the ● team (say), tells us two numbers--at this stage, two whole numbers (which may include zero.)

6) We graph the point which the 1st player named.

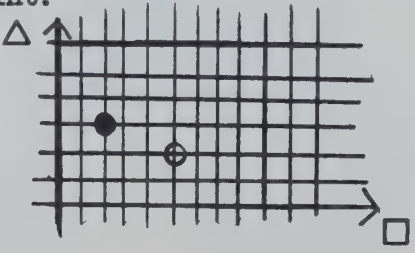
(Suppose it was (2, 3).)



7) The first player for the other team tells us two numbers.

(Let us suppose he says (5, 2).)

8) We graph this point:



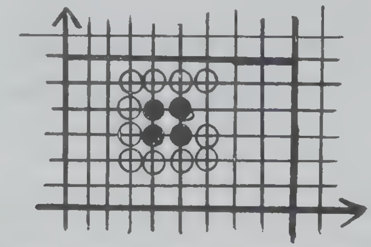
9) We continue playing in this way.

Some remarks:

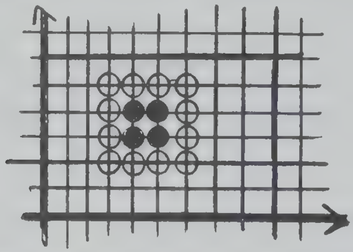
10) You can stop whenever you wish (say, at the end of the period). The team with more points on the graph wins.

11) The basic rule, of course, is that when a point (or a group of points) is surrounded by the opponent's points we erase the surrounding points.

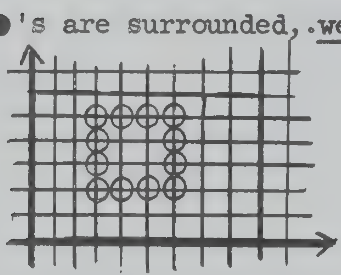
Thus, suppose we had this pattern;



Suppose the next player for the ○ team said "six, four." We would mark a ○ at (6, 4):



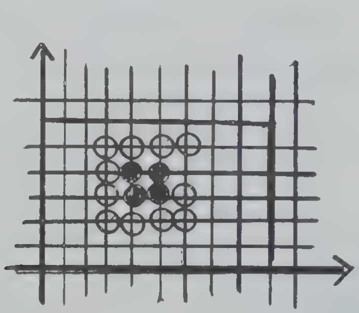
Now--since the four ●'s are surrounded, we erase them!



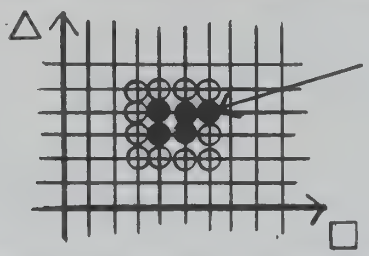
This, of course, was a victory for the ○ team!

12) If possible, of course, a team moves in such a way as to defend itself against possible surrounding:

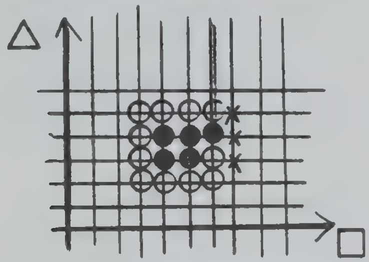
Suppose, for example, the situation looks like this, and it is the ● team's turn to move.



In this case, the ● team should defend itself by playing "six, four." The result would be:

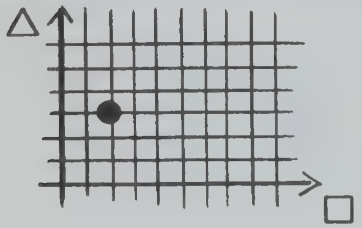


The ● team is no longer in danger of being surrounded: they have connections to three vacant points, namely (7, 3), (7, 4), and (7, 5):

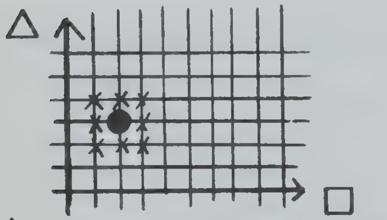


The ● 's have connections to three vacant points; consequently they are not surrounded.

13) We count connections along graph lines, and also along the diagonals.



This point is said to be "connected" to (1, 2), (1, 3), (1, 4), (2, 4), (2, 2), (3, 4), (3, 3), and (3, 2):

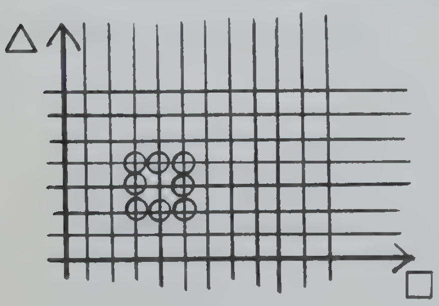


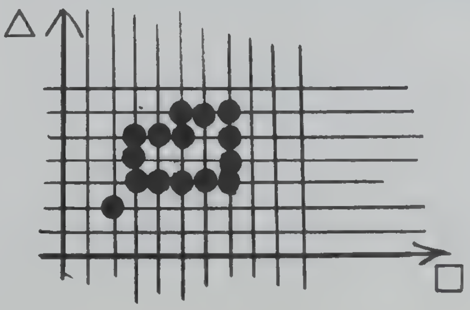
i) If any of these eight points are vacant, the point is not surrounded.

ii) If all eight of these points are occupied by enemy spots, then the at (2, 3) is surrounded and must be erased.

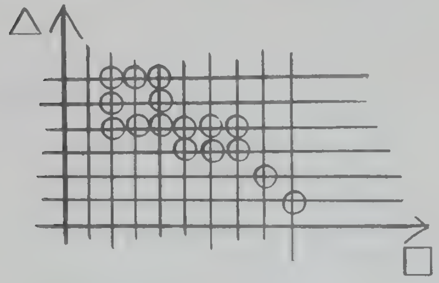
14) A vacant spot or spots entirely surrounded by one team is called an "eye."

These are both eyes:

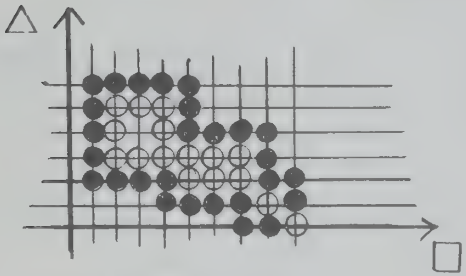




15) It is a rule of the game that all points connected to an eye are safe, and cannot be surrounded.



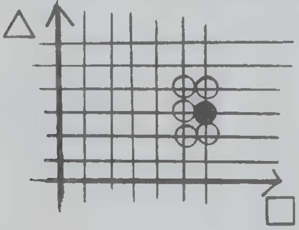
All of these points are "safe"-- they are connected to an "eye"!



The ○ points are not surrounded, because they have an "eye" at (3, 4).

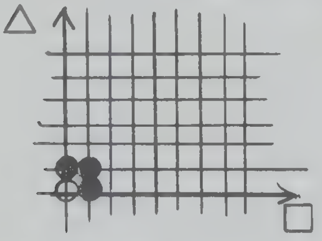
16) If, in the example above, the ○ team should unwisely fill in the eye at (3, 4), then all of their points would be surrounded, and would be erased!

17) A point on an edge can be surrounded by 5 opposing points:



The ● at (6, 3) is surrounded and must now be erased.

18) A point in a corner can be surrounded by 3 opposing points:



The ○ at (0, 0) is surrounded and must be erased.

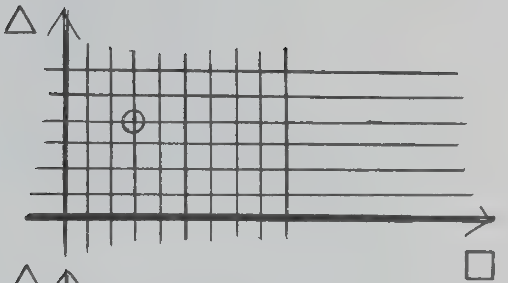
19) If a player calls for a point already occupied, that move is illegal. We make no mark, and go on to the next player on the other team. (In other words, an illegal move is simply wasted. It is a victory of one point for the opposing team.)

Similarly, if a player calls for a point outside of the boundary, the move is illegal and is wasted.

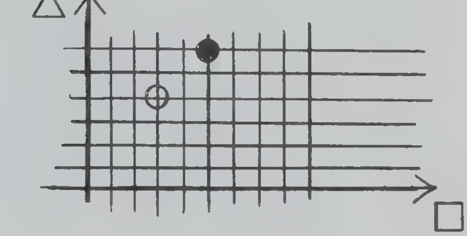
Example:

A ○
B ●

Team A: (3, 4)

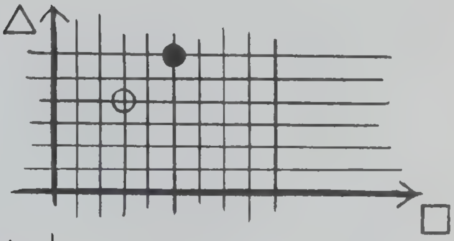


Team B: (5, 6)

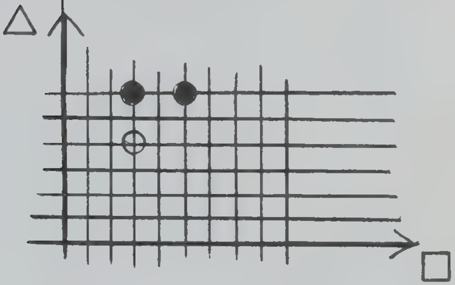


Team A: (6, 7)

ILLEGAL!!

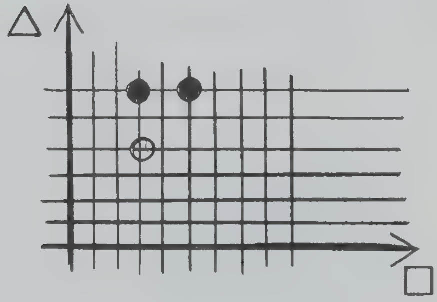


Team B: (3, 6)

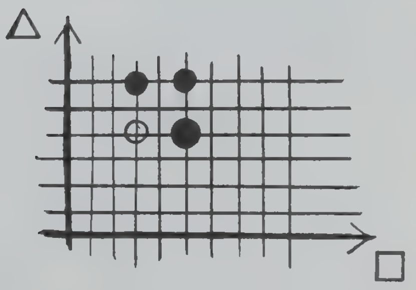


Team A: (5, 6)

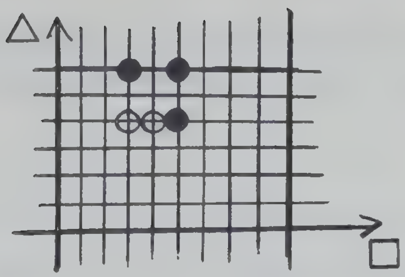
ILLEGAL!!



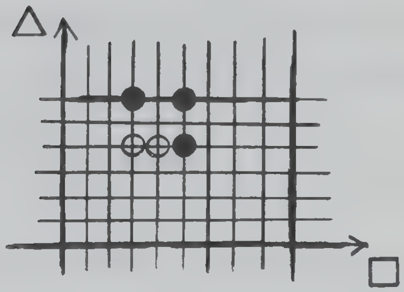
Team B: (5, 4)



Team A: (4, 4)

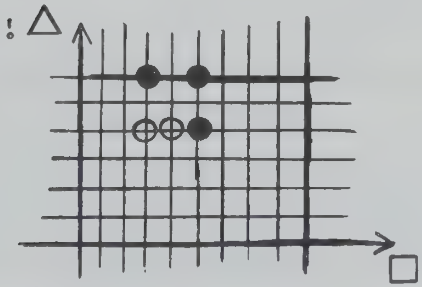


Team B: (5, 5)

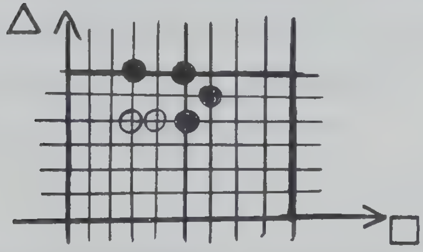


Team A: (3, 4)

ILLEGAL!!

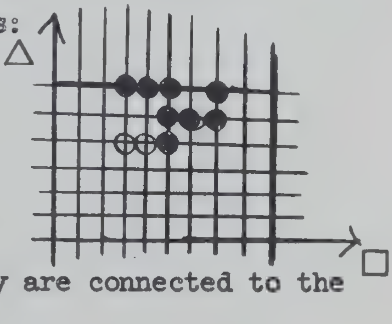


Team B: (6, 5)



What is happening here is this: Team A's mistakes have cost them 3 points. Team B has 5 points on the graph, compared with Team A's two points.

Notice that if Team B is able to get these points:



. . . then all of Team B's points are safe, since they are connected to the "eye" at (6, 6)!!

20) We usually take the first pair of numbers each student calls out! We do not permit him to change it. This causes students to become quite careful in choosing their points.

Now--why do we consider this game important?

1) It gives students a pleasant form of practice in plotting points.

2) The students enjoy it--hence it adds to their pleasure in studying mathematics. It is very important in mathematics for the subject to be fun!

3) The students enjoy the social interplay in choosing up teams. This gives us a chance to get every student participating.

4) It gets the student thinking! He cannot win a game like this unless he thinks.

Also, he cannot do mathematics unless he thinks. We can only teach math to shrewd students.

"It is said that there was once a doctor who could only cure fits, so he first threw his patients into fits, and then he cured them."

We can only teach math to shrewd students. So we let them play games. In order to win, they become shrewd. Then we can teach them mathematics!

5) The fifth reason is that we enjoy this game ourselves. We hope that, once you have become familiar with the rules, you, too, will get fun out of this game, and will convey this pleasure to your students.

We hope you will learn to enjoy this game as much as we do.

LESSON IV -- CHAPTER IV

Although we have called this lesson "Review," it does introduce one new idea--the idea of a solution set we are never finished listing. For example, in problem 26, page 26, we are asked to write the truth set for the open sentence

26) $1 < 3 + (2 \times \square) < 10$,

using whole numbers and fractions.

Evidently 0, 1, 2, and 3 belong to the truth set:

{0, 1, 2, 3, ____________}

But this is not all. The numbers 1/2, 1 1/2, 2 1/2, and 3 1/2 also belong:

{0, 1, 2, 3, 1/2, 1 1/2, 2 1/2, 3 1/2, _____________}

This is still not all! 1/3, 1 1/3, 2 1/3, and 3 1/3 also belong to the truth set: {0, 1, 2, 3, 1/2, 1 1/2, 2 1/2, 3 1/2, 1/3, 1 1/3, 2 1/3, 3 1/3, _____}

This is still not all! 2/3, 3 1/7, 2 1/10, 3 2/31, and 1 3/8 also belong to the truth set:

{0, 1, 2, 3, 1/2, 1/3, 1 1/2, 3 1/3, 2 1/3, 2 1/2, 3 1/2, 1 1/3, 3 1/7, 2/3, 2 1/10, 3 2/31, 1 3/8, _____________________}

Clearly, this is still not the complete list! We have left out 1 2/3, 2 2/3, 3 2/3, 1/27, 3/8, 2 3/8, 1/61, 1 19/23, 1/1959, 1/1066, 2 1/1732, and 3 3/10. Nor are these the only numbers that we have left out. It is clear that out list will never be complete.

In such a case, the mathematician indicates that the list will never be complete by writing three dots,

{0, 1, 2, 3, 1/2, 1 1/2, 2 1/2, 3 1/2, 1/3, 1 1/3, 2 1/3, 3 1/3, 1 3/8, 2/3, 3 1/7, 2 1/10, 3 2/31, . . .}

The final three dots indicate that we realize that the list is incomplete, but we know that it connot be made complete.

(pages 23 - 24)

1) $2 + 7 = 9$ True

2) False

3) True

4) True

5) Open

6) . . . eitner true or false, depending on who sits in the front row in your class.

7) Open

8) . . . same as number six -- depends on your class.

9) Open

10) Open

11) Open

12) False

Can you find the truth set . . .

13) . . . depends on your class. Might be something like this:

{Shirley, Jane, Marie, Tom, Jerry, Bob}

14) {7}

15) {7}

16) {2 1/2}

17) {1 1/3}

18) {2 1/3}

19) {3 2/3}

20) {3}

21) {1}

Can you make a table . . .

22) . . . depends on your class. Might be something like this:

23) See next page

24) See next page

25) {0, 1, 2, 3}

26) {0, 1, 2, 3, 1/2, 1 1/2, 2 1/2, 3 1/2, 1/3, 1 1/3, 2 17/19,. . .}

(Notice the three dots!! Compare the discussion at the beginning of this chapter in the Teacher's Manual.)

"__________"	(______________)
Shirley	1st
Jane	1st
Doris	3rd
Wilbur	4th
Evelyn	5th
Marie	1st
Tom	1st
Albert	2nd
Ruth	5th
Jerry	1st
Larry	4th

and so on. . .

#23

□	△
0	5
1	6
2	7
3	8
4	9
1/2	5 1/2
1 1/2	6 1/2
1/3	5 1/3
2 1/3	7 1/3
⋮	⋮

#24

□	△
0	18
1	16
2	14
3	12
1/2	17
1 1/2	15
1/3	17 1/3
3 1/2	11
3/4	16 1/2
⋮	⋮

27) $3 < 1 + (2 \times \square) < 12$ Whole numbers

Ans. $\{2, 3, 4, 5\}$ . . . . Note that "$3 < 3$" is false,
3 is not less than 3

28) Whole numbers and fractions Ans. $\{2, 3, 4, 5, 1\ 1/2, 2\ 1/2, 2\ 3/4, 4\ 7/8, 3\ 1/3, 1\ 2/3, 5\ 1/4, \ldots\}$

29) Ans. $\{3, 4\}$

30) Ans. $\{3, 4, 2\ 1/2, 2\ 1/4, 2\ 1/8, 2\ 1/16, 3\ 7/8, 3\ 2/3, \ldots\}$

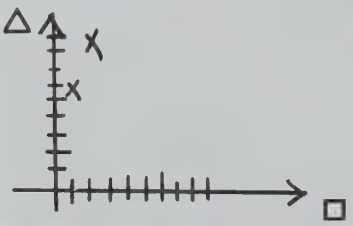
31)
32) On graph paper Concerning problems 31-36 we have two remarks:

1) We hope some more students can discover for themselves the pattern in these problems (which, of course, is actually the concept of slope of a straight line).

Notice that in problem 35, for example, we might begin with the point

□	△
1	6

. We can now find further points by using the pattern "over one, up 3".



Using this pattern, we get (as our third point) (3, 12). If we now substitute 3 in the □, and 12 in the △, the statement is true! We found this point without needing to calculate!

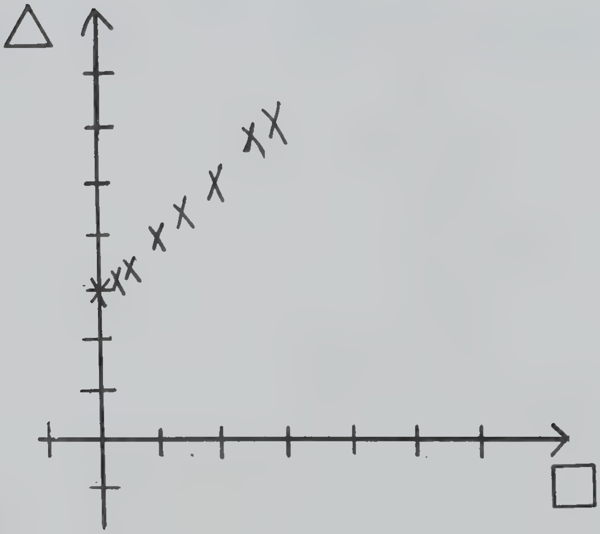
2) You may wish to encourage some of your students to use fractions as well as whole numbers. If they do, then problem 31, (for example), might look something like this:

31) △ = □ + 3

Table:

□	△
0	3
1	4
2	5
3	6
1/2	3 1/2
1/4	3 1/4
1 1/2	4 1/2
2 2/3	5 2/3

Graph:



△ = □ + 3

LESSON V - CHAPTER V

We hope you have gotten used to two aspects of the Madison Project:

1) In each lesson we try to introduce something new, and also to review some of the earlier work. We do not expect a topic to be mastered when it is first presented; most students will master it gradually, during its subsequent re-appearances as a review topic.

2) The "Light Touch": We do not necessarily try to cover all of a lesson. If we cannot, we use our own judgment in deciding whether to continue with that lesson on a subsequent day, or whether to let the matter drop. After all, there is plenty of time -- we don't have to learn everything right away. It is best to skip over a topic if this will help to maintain really lively class discussions.

(page 29) Work with tables and graphs.
Also show some solutions with fractions.

LESSON 6 - CHAPTER 6

Our students usually find this a particularly exciting lesson.

For one thing, it gives them a chance to argue with one another. We encourage this. We often take a vote of the class to see how many think the sentence is an identity, and how many think it is not an identity.

In order to prove that a statement is not an identity, the students must find some numbers which, when correctly substituted, produce a false statement.

Class discussion on problem 1) might perhaps go something like this:

You: Do you think #1 is an identity?

John: No, because (3, 2) won't work.

You: Is that right? (Here we would make a special effort to sound like we agreed with whatever the students tell us. We are careful to avoid sounding sarcastic, dogmatic, authoritarian, or unpleasant. We do not want an atmosphere where we seem to pass judgment on the students. Actually, we don't need to voice our judgments anyhow, -- there is always some other student who is ready to point out any errors. We just go along in our bemused -- but pleasant! --way. Of course, if the truth be known, we probably have out little tricks for steering the conversation, but we hope that nobody notices . . .)

Gerry: No, $2 + 3 = 5$, and $3 + 2 = 5$, so it is an identity.

Marie: It is an identity!

John: Oh, I made a mistake. . .

You: Let's take a vote: how many think it is an identity?

Perhaps 2/3 of the class raise their hands.

Who thinks it is not an identity?

Most of the other students raise their hands.

Mary, can you prove it?

. . . and so on . . .

Now, several exciting things are likely to happen:

1) Somebody is likely to find a substitution that does work, and conclude that the statement is an identity.

This is not a correct proof! To prove that a statement is an identity, we must prove not that one substitution yields a true statement, but that every

correct substitution does.

We can either challenge this false proof ourselves, or we can try to maneuver the conversation so that some student will point out the fallacy.

2) In the case of a statement that *is* an identity, some student is likely to become convinced that it is not, and to go to great lengths to find a correct substitution which will yield a false statement. You and I know he will never find one, because there isn't any.

But we don't tell him this!!

It's his job to decide whether or not the statement is an identity, and if he is determined to find a case to prove that it isn't, we're not going to hold him back. We let him try, and after every few trials we take a vote of the class, to see how many think that he will someday find a substitution that will make the statement false.

When we think it's time to go on, we leave the issue undecided (if necessary) and try a new problem.

Answers: 1) $\square + \triangle = \triangle + \square$

2) $\square \times \triangle = \triangle \times \square$

These *are* identities. Every correctly made substitution will produce a *true* statement.

3) $\square + \triangle = 15$ This is *not* an identity. For example, put 3 in the square and 4 in the triangle, The result is:

$$\boxed{3} + \overset{4}{\triangle} = 15 \qquad 7 \neq 15$$

even though we substituted correctly when we put 3 in the $\square$ and 4 in the $\triangle$.

This gives you an opportunity to review *truth sets*. The substitution $10 \rightarrow \square$, $5 \rightarrow \triangle$, produces a true statement. Hence (10, 5) is an element of the truth set.

However, (3, 4) is *not* an element of the truth set.

If $\square + \triangle = 15$ *had* been an identity, there would be *no* element (of the proper form) which did *not* belong to the truth set.

4) This *is* an identity. Incidentally, if the students substitute numbers so that they encounter negative numbers (which they have probably not yet studied), our suggestion is:

Don't waste time on this now!

Instead, suggest to the class that we will learn about "numbers like that" later on.

5) This is not an identity. (Again, be prepared to skip over negative numbers, as we did in problem 4, if the difficulty should arise.)

6) This is an identity. The students cannot prove this one either way: they can't find numbers to make it false, because there aren't any; and they can't show that every set of numbers, properly substituted, will make it true, because there are infinitely many numbers, and the students can't possibly substitute all numbers. But after a great many student trials, we can take a vote, and see how many students think that this really is an identity.

7) Not an identity.

8) This is an identity, and a very important one--but it may be best not to tell the students so. We hope that, after a while, they will come to recognize this important identity, but it may be wiser not to emphasize it too much just yet.

9) An identity. (Actually, the same as 8.)

10) An identity. (Actually, the same as 8 and 9.)

11) An identity. (A generalization from 8, 9, and 10.)

12) An identity. (Still this same one.)

13) An identity. (This is still the same; we hope the students are beginning to suspect as much, at least vaguely.)

14) Not an identity.

15) An identity. Students can't prove this, because we have not yet provided them with a logic and set of axioms. Frequently, however, some bright student tries to "explain" why this must be an identity. Our response to such a "proof" is to be non-commital.

16) An identity.

17) An identity.

18) An identity. (Our old friend, again.)

19) Not an identity.

20) An identity.

21) An identity. (Our old friend, in still another form.)

22) No. In order to prove that this sentence is an identity, John would have to show that every substitution yields a true sentence, and not merely that $1 \rightarrow \square$ and $1 \rightarrow \triangle$ does.

23) Yes.

24) In fact, the sentence is not an identity. This is easy to prove. We need merely find a correct substitution that will produce a false statement as a result.

Substitute 0 → □, 1 → △ . The result is: $(3+1) \times 0 = (0+3) \times 1$

$0 = 3$, which is false.

Since this correct substitution has resulted in a false statement, the original open sentence cannot be an identity.

LESSON 7 - CHAPTER 7

It is a big help here to use a very small sign, written very high, to denote a positive 3, i. e.,

$$^+3$$

in contrast to a large sign, centered on the line, to denote addition:

$$2+3$$

Remember that what we are dealing with here are "numbers that came with signs attached to them." When we add two signed numbers, we are dealing with something like this:

$$^+2+{}^-3$$

In most cases, if no sign is indicated,

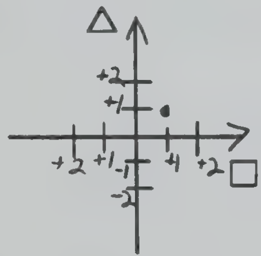
e. g. 2

when what is meant is a positive sign:

meaning: $^+2$

1) A few uses of numbers that have signs attached to them:
    - i) Indicating up vs. down on a graph, or right vs. left;
    - ii) Indicating gains vs. losses in a game, or in a business transaction;
    - iii) Indicating temperatures above zero vs. temperatures below zero;
    - iv) Indicating altitudes above sea level vs. altitudes below sea level.
2) Ruth, of course, is right.

3)



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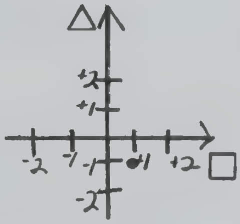

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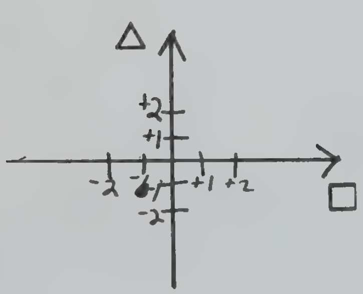

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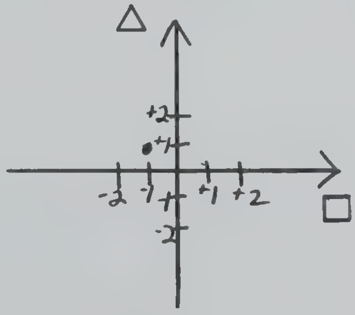


7-14, etc.

LESSON 8 - CHAPTER 8

Answers:

1) Using a single signed number (as suggested in the workbook), we would say that the score is $^+3$.

2) One run scored for Boston, so the score would be $^+3$.

3) The score would be $^+1$ (two more Boston runs came in on this play.)

4) The score would now be $^-1$ (i. e., Boston is now one run ahead).

5) Since Boston picks up two more runs, the score at this point would be $^-3$ (that is, Boston is now three runs ahead).

THE MATRIX GAME

A game which we use in all our classes might well be introduced at this point.

To play, we divide the class into two teams (we usually let the students themselves choose up sides, which seems to add to their enjoyment). Call one team the "Green" team, and the other the "Red" team. The first player for the Green team will write either "A", "B", or "C" on a piece of paper; simultaneously, the 1st player for the Red team will write either "L", "M", of "P" on a piece of paper. Neither player knows, at this stage, what the other player has written.

Before beginning play, wr wrote on the blackboard this matrix:

$$\begin{array}{c} \\ L \\ M \\ P \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \left(\begin{array}{ccc} ^+1 & ^-1 & 0 \\ ^-2 & 0 & ^+2 \\ ^+1 & ^+1 & ^-2 \end{array} \right) \end{array}$$

Green's choices

Positive numbers denote victory for Green. Negative numbers denote victory for Red.

Red's choices

The score is $\boxed{\quad}$

Diagram 1. Matrix and Accompanying Notes, As We Write Them on the Board Before Beginning to Play the Matrix Game.

The first player for each team has written his choice. These choices are now read (by two student referees, perhaps).

Suppose the choices were:

Green: "B"

Red: "P"

We find the outcome by reading the matrix:

$$\begin{array}{c|ccc}
 & A & B & C \\
\hline
L & ^{+}1 & ^{-}1 & 0 \\
M & ^{-}2 & 0 & ^{+}2 \\
P & ^{+}1 & \boxed{^{+}1} & ^{-}2
\end{array}$$

Since Green chose B, we read the 2nd column.

Since Red chose P, we read the 3rd row.

The outcome is the entry in the 3rd row, 2nd column.

The score is now $\boxed{^{+}1}$ -- that is, Green is ahead by one point.

To check your understanding, suppose on the next play the 2nd Green player chose "A", and the 2nd Red player chose "M". The result of this play would be;

$$\begin{array}{c|ccc}
 & A & B & C \\
\hline
L & ^{+}1 & ^{-}1 & 0 \\
M & \boxed{^{-}2} & 0 & ^{+}2 \\
P & ^{+}1 & ^{+}1 & ^{-}2
\end{array}$$

Consequently, the reslut of this play is $^{-}2$ (that is, a 2-poing victory for Red).

Before this play, the score was $\boxed{^{+}1}$ (Green ahead by one point).

On this play, the result was $^{-}2$ (Red won a 2-point victory).

Hence, the cumulative score is: $\boxed{^{-}1}$ (Red is now one point ahead).

We usually stop the game at the end of the period, or at some convenient time; whoever is then ahead is declared the winner.

CHAPTER 9 - LESSONS 9, 10, etc.

You will probably want to use this "Postman" example for a month or longer, whenever you have occasion to add or subtract signed numbers.

Answers: 2) $^{+}2 + {}^{+}4 = ?$ The postman brings you a check for two dollars. $^{+}2$

and he also brings you $^{+}2 +$

a check $^{+}2 + {}^{+}$

for four dollars $^{+}2 + {}^{+}4$

As a result, you are richer by an amount of six dollars: $^{+}2 + {}^{+}4 = {}^{+}6$

3) $^{+}5 + {}^{-}2 = ?$

The postman brings you a check for five dollars $^{+}5$

and he also brings you $^{+}5 +$

a bill $^{+}5 + {}^{-}$

for two dollars. $^{+}5 + {}^{-}2$

How much better off (or worse off) are you? $^{+}5 + {}^{-}2 = ?$

You are better off by an amount of three dollars $^{+}5 + {}^{-}2 = {}^{+}3$

4) Brings bill for $2 and bill for $3. $^{-}2 + {}^{-}3 = {}^{-}5$

5) Brings check for $5 and bill for $6. $^{+}5 + {}^{-}6 = {}^{-}1$

6) Brings bill for $7, check for $9. $^{-}7 + {}^{+}9 = {}^{+}2$

7) Brings bill for $5, check for $1. $^{-}5 + {}^{+}1 = {}^{-}4$

8) Brings bill for $3, plus (say) a letter with no check or bill at all (perhaps a birthday card or an advertisement). $^{-}3 + 0 = {}^{-}3$

9) Brings bill for $2, and bill for $5. $^{-}2 + {}^{-}5 = {}^{-}7$

10) Brings check for $6, plus check for $3. $^{+}6 + {}^{+}3 = {}^{+}9$

11) Brings check for $8, plus bill for $1. $^{+}8 + {}^{-}1 = {}^{+}7$

12) Brings bill for $3, plus bill for $6. $^{-}3 + {}^{-}6 = {}^{-}9$

13) Brings check for $1, plus check for $12. $^{+}1 + {}^{+}12 = {}^{+}13$

14) Brings check for $7, plus bill for $9. $^{+}7 + {}^{-}9 = {}^{-}2$

15) Brings check for $2, plus check for $17. $^{+}2 + {}^{+}17 = {}^{+}19$

16) We can explain subtraction by saying that the postman "takes" away a check," or "takes away a letter."

If you need to go into more detail, we can agree that we will total up all checks and bills every day, without stopping to verify that they are really for us. On some days, the postman comes and says: "That check I brought yesterday -- it was really for the woman upstairs. Let me have it and I'll give it to her." In such a case we say the postman takes away a check; since we had included this check in yesterday's daily total, we must now decrease our recorded balance by the amount of this check.

For ${}^{+}7 - {}^{+}2 = ?$

we would say:

The postman brings a check for \$7 (which makes us \$7 richer), and takes away a check for \$2 (which makes us \$2 poorer, as explained above).

17) Brings check for \$9, and takes away a check for \$2. ${}^{+}9 - {}^{+}2 = {}^{+}7$

18) Brings check for \$15, and takes away a check for \$5. ${}^{+}15 - {}^{+}5 = {}^{+}10$

19) Brings check for \$8, and takes away a check for \$2. ${}^{+}8 - {}^{+}2 = {}^{+}6$

20) Brings bill for \$5, and takes away a bill for \$1. ${}^{-}5 - {}^{-}1 = {}^{-}4$

21) Brings bill for \$5, and brings also a bill for \$1. ${}^{-}5 + {}^{-}1 = {}^{-}6$

22) Brings check for \$10, and takes away a check for \$2. ${}^{+}10 - {}^{+}2 = {}^{+}8$

23) Brings check for \$8, and takes away a check for \$7. Ans. $\{{}^{+}1\}$

24) Brings bill for \$5, and takes away a bill for \$2. Ans. $\{{}^{-}3\}$

25) Ans. ${}^{-}7$

26) ${}^{-}9$

27) Brings check for \$7, and takes away a bill for \$2. Each of these actions makes us richer. ${}^{+}7 - {}^{-}2 = {}^{+}9$

28) ${}^{+}7 - {}^{-}4 = 11$ Some bright students may notice that "taking away a bill for \$2" is economically the same thing as "bringing a check for \$2;" in one class this became known as "Cecile's Rule", after the girl who first stated it.

Using "Cecile's Rule," she would say:

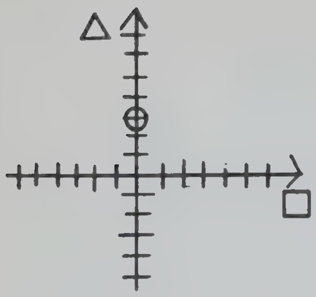
$${}^{+}7 - {}^{-}4 = {}^{+}7 + {}^{+}4 = {}^{+}11$$

But we don't tell this rule to the students--we wait until they discover it (which probably won't happen as early as this lesson).

29) ${}^{+}8$ 30) ${}^{-}8$ 31) ${}^{-}6$

32) ${}^{-}7$ 33) ${}^{-}6$

34)



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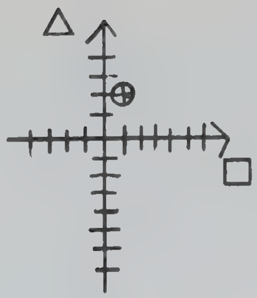

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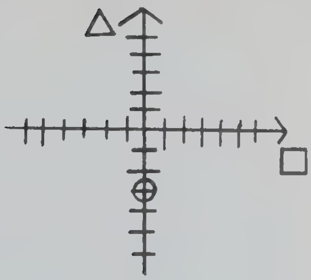

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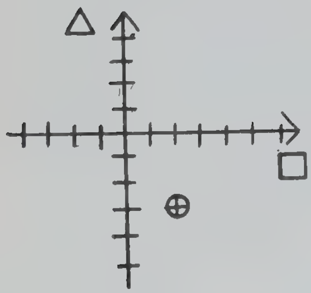

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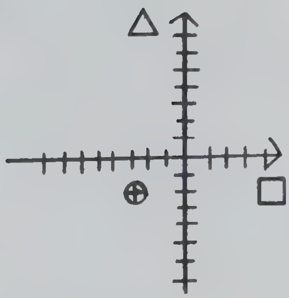

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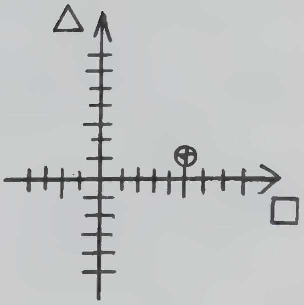

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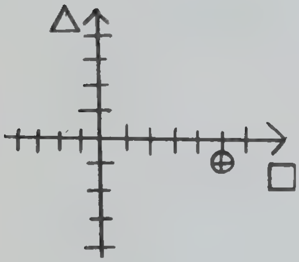

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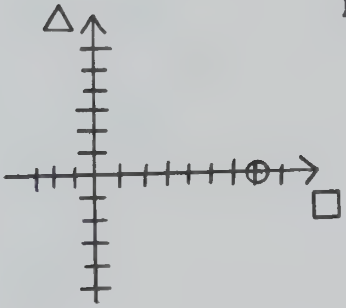

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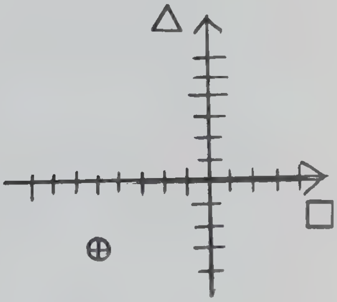

43) $\{^+5\}$ (or, of course, we could write this simply $\{5\}$).
44) $\{^-2\}$
45) $\{^-3\}$
46) $\{^+2\}$
47) $\{0\}$
48) $\{^-2\}$

LESSON 10 - CHAPTERS 10 and 11

The main thing here is the light touch.

We do not usually spend more than 10 to 20 minutes on Chapter 10 during this lesson. We do, however, return to this material frequently --and "casually"--in subsequent lessons.

What usually happens. All students quickly learn how to substitute and test for roots by trial-and-error.

A few usually discover the "two secrets": the product of the roots is the last coefficient in the equation, and the sum of the roots is the second coefficient.

For example

1) $(\square \times \square) - (13 \times \square) + 22 = 0$

The truth set is: $\{2, 11\}$

$2 + 11 = 13$

$2 \times 11 = 22$

2) $(\square \times \square) - (15 \times \square) + 44 = 0$

The truth set is: $\{4, 11\}$

$4 \times 11 = 44$

$4 + 11 = 15$

A few students usually discover these "secrets". When they do, they can use their discovery to solve future equations faster than we can write them down on the board. But we encourage them to protect their "secret", and not let the other students find out how they are doing it.

This is usually one of our most exciting and gratifying lessons.

To make the "secret" work easily, most equations have roots that are prime. This includes problems 1 - 5, 7, 9, 12, 14, 15.

Those equations with roots that are not prime are somewhat more difficult (as in the case of 6, 8, and 17), or even quite a bit more difficult (10, 11, 16, and 19).

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You can sometimes induce the class to discover the "secrets" more quickly by repeatedly asking them if they have yet found the secret, or even good-naturedly chiding them if they have not.

Answers:

1)	{2, 11}	usually easy
2)	{4, 11}	" "
3)	{3, 11}	" "
4)	{5, 3}	" "
5)	{8, 2}	" "
6)	{6, 4}	slightly harder
7)	{7, 3}	easy
8)	{6, 1}	usually slightly harder
9)	{2, 1}	usually easy
10)	{12, 9}	hard
11)	{20, 4}	usually hard
12)	{9, 11}	easy
13)		
14)	{5, 2}	easy
15)	{7, 2}	easy
16)	{16, 9}	hard
17)	{8, 10}	fairly easy
19)	{11, 32}	hard!!

LESSON 10 - CHAPTER 11

How Do We Decide Questions in Mathematics?

We do not usually spend an entire lesson on our first treatment of quadratic equations. You might finish the day's lesson by beginning the work of Chapter 11.

If the postman example has settled the question of the arithmetic of signed numbers, you may prefer to skip Chapters 11, 12, and 13.

Alternatively, you can omit Chapter 9, and use Chapters 11, 12, and 13 to develop the arithmetic of signed numbers.

The point of the discussion on experiments is this: A statement like ${}^+2x + {}^+3 = {}^+6$ is an abstract statement. Before we can "perform an experiment," we must work out an interpretation. That is, we must identify each abstract concept $[{}^+2, {}^+3, {}^+6, \text{ and } x]$ with something in the real world. We develop such an interpretation in the next chapter.

CHAPTER 12

It is important with this lesson to get a lively class discussion if that is possible.

For $^{-}2 \times$ ___ , we say that we wish to undo, or cancel out, something . . . and to "undo" it twice.

Consequently, we could say that

$$^{-}2 \times {}^{+}3$$

means we had recorded earnings of $3 on two occasions, but (perhaps because we received bad checks) we now must reverse ourselves and "undo" this. That is, we must subtract $6 from the balance shown on our books.

Similarly, we would say that

$$^{-}2 \times {}^{-}3$$

means:

$^{-}3$ a loss of $3

$^{-}2 \times$ to be undone twice

(perhaps someone is making good the two bad checks).

To undo this loss, we must add $6 to the balance shown on our books.

We do not insist that students accept this interpretation--we leave it up to them to decide whether they are convinced by the accumulation of evidence. (If they don't like this evidence, we have some more coming up in the next chapter.)

We start a list in class:

Satisfactory	Doubtful
$^{+}2 + {}^{+}3 = {}^{+}5$	$^{-}2 \times {}^{-}3$
$\underset{\substack{\uparrow \\ \text{represents} \\ \underline{\text{twice}}}}{{}^{+}2} \times \underset{\substack{\nwarrow \\ \text{represents} \\ \text{money} \\ \text{earned}}}{{}^{+}3} = {}^{+}6$	Evidence: 1) Interpretation using gains and losses 2) Identities 3) Graphs

In the left column we list problems (representing types) that are acceptable to the class at present. (They may be removed later if people become skeptical about them.)

In the right column we list problems which are still doubtful, and under each problem we list the evidence that is available to settle the question. (The evidence from "identities" and "graphs" will be presented in subsequent chapters.)

CHAPTER 13

This is normally a single lesson.

Purposes: Review of preceding lesson (Chapter 12)
Introduction of the concept of "unhappening"
Use of commutative law as an alternative approach to "$^{-}2 \times {}^{+}3 =$?"
Use of graphs as an alternative approach to "$^{+}3 \times {}^{-}1 =$?" and "$^{+}3 \times {}^{-}2 =$?"

CHAPTER 14

This is the first lesson in a sequence of lessons devoted to the idea of identities.

We hope the sutdents will learn:

1) What an identity is. (I. e., *every* substitution produces a *true* statement)

2) Recognition of a few of the more fundamental identities, especially:

$$\square \times \triangle = \triangle \times \square$$
$$\square + \triangle = \triangle + \square$$
$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$
$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$$
$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$
$$\square \cdot 1 = \square$$

3) Some insight into the *form* of various identities. (For example, the similarity of

$\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$ and

$\square \times (\triangle + 7) = (\square \times \triangle) + (\square \times 7).$)

4) The idea that *two* identities can often be combined to produce a *third* identity. E. g.,

i) $\square + \triangle = \triangle + \square$

ii) $(\square + 2) \times (\square + 2) = (\square \times \square) + (4 \times \square) + 4$

Combining i) and ii), we can get:

iii) $(2 + \square) \times (\square + 2) = (\square \times \square) + (4 \times \square) + 4$

5) There is an impossibly long list of correct identities, including:

i) $\square + \triangle = \triangle + \square$

ii) $\square \times \triangle = \triangle \times \square$

iii) $(\square + 2) \times (\square + 2) = (\square \times \square) + (4 \times \square) + 4$

iv) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$

v) $\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$

vi) $\square \times (3 + \triangle) = \square \times (\triangle + 3)$

vii) $\square \times (3 + \triangle) = (3 + \triangle) \times \square$

viii) $\square \times (3 + \triangle) = (\triangle + 3) \times \square$

. . . and so on

6) We can replace this list by a much shorter list. How? By using the idea of *implication*. For example (cf. list above):

We can eliminate vi), since it is a consequence of i).

We can eliminate vii) from the list, since ii) implies vii). (We would write this ii) $\Rightarrow$ vii).)

We can eliminate viii), since i) and ii) *together* imply viii). (We would write this i) and ii) $\Rightarrow$ viii).)

We can eliminate v) from the list--it tells us nothing new beyond what iv) tells us.

7) Finally, there is the question of *how far can we get the students to go with this*. The *idea* itself culminates with the concept of a set of *axioms*, and with the "proof" of a "theorem" such as:

$$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + \left[2 \times (\square \times \triangle)\right] + (\triangle \times \triangle)$$

We do not insist on getting this far with the student during the "1st Course". (We shall return to this sequence of ideas again later in "2nd Course".)

CHAPTERS 15 and 16

These two chapters continue the sequence on identities and implication.

CHAPTER 17

The main topics of "lst Course" are these four:

i) Equations (or "open sentences")
ii) Identities (and Implication)
iii) Graphs
iv) Signed Numbers

This Chapter is aimed at consolidating our progress in working with these four topics.

CHAPTER 18 ("Derivations")

I should be ready for anything by now, but this lesson astonished me the first time I taught it. It was a tremendous success with the class (who were sixth-graders), and was easily completed in one one-hour period. At the end, the class was eager to continue.

This lesson introduces several new notations and words. Personally, I treat these very casually, and the class accepts them without question. In subsequent lessons some questions may arise--but by then we will have enough experience on which to build a good answer.

One example: List #III introduces the first use of letters as placeholders.

I do not explain this.

I write 1, 2, and 3 of List #III.

Then I casually ask the class "Do you mind if I use letters?" They have always assented eagerly (they have seen letters used in other books). So then I write, without advance explanation,

$$(a+b)\times(c+d)=(a\times c)+(a\times d)+(b\times c)+(b\times d)$$

If any question is raised at this point, I simply say that "a" is a placeholder just like "$\square$" or "$\triangle$".

Personally, I think you can handle this lesson better if you do not use the workbook during the lesson. You yourself can read the workbook in advance, and then unobtrusively lead the class discussion to follow the general lines indicated in the workbook.

Answers: 1) The list can be shortened. Identities 1) and 2) imply Identity 3).

How? Here is the derivation of #3) :

2) $\square\times(\triangle+\nabla)=(\square\times\triangle)+(\square\times\nabla)$ ⟵ We start here

$\underline{\qquad\qquad\qquad}$ ↓

$\underline{\qquad\qquad\qquad}$ and we wish to get here ↙

3) $\square\times(\triangle+\nabla)=(\triangle\times\square)+(\square\times\nabla)$

The idea is to make only one change at each step, and to justify each change by using some identity on the list.

Evidently, we can get from 2) to 3) in a single step, by using identity 1) to permute (or "flip," or "reverse") the factors $\square$ and $\triangle$ in the first parentheses on the right:

#1) 2) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$
3) $\square \times (\triangle + \nabla) = (\triangle \times \square) + (\square \times \nabla)$

2) Yes. cf. answer to question 1.
3) Yes. " " " " "
4) I have found that this word works very well with my students. It seems to help a great deal to use intuitive language, rather than "formal language".

List #II

5) Yes. Identities 1), and 2), and 3) $\Longrightarrow$ Identity 4) .
6) Here is one possible derivation of Identity 4) .

2) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$
#1 i) $\square \times (\triangle + \nabla) = (\triangle \times \square) + (\square \times \nabla)$
#1 ii) $\square \times (\triangle + \nabla) = (\triangle \times \square) + (\nabla \times \square)$
#3 iii) $\square \times (\triangle + \nabla) = (\nabla \times \square) + (\triangle \times \square)$

Notes: 1) We get from 2) to i) by "flipping" on this " " sign:

$$\square \times (\triangle + \nabla) = (\square \underset{\uparrow}{\times} \triangle) + (\square \times \nabla)$$

This is justifies by Identity #1.

2) We get from i) to ii) by "flipping" on this " " sign:

$$\square \times (\triangle + \nabla) = (\triangle \times \square) + (\nabla \underset{\uparrow}{\times} \square)$$

This is justified by Identity #1.

Remember, in Identity 1) we can substitute *any* number into the $\Box$, and *any* number into the $\triangle$, so Identity 1) tells us that, *in any product*, we can "flip" the two factors.

3) We get from ii) to iii) by "flipping" on this "+" sign:

$$\Box \times (\triangle + \nabla) = (\triangle \times \Box) \underset{\uparrow}{+} (\nabla \times \Box)$$

This is justified by Identity 3), which says that in *any* sum,

$$\boxed{(\triangle \times \Box)} + \boxed{(\nabla \times \Box)}$$

we may reverse the order of the two terms:

$$\boxed{(\nabla \times \Box)} + \boxed{(\triangle \times \Box)}$$

4) In making derivations, it is wise to look ahead and see where we would like to go. I sometimes mark terms that we like with a smiling face ☺ , and for terms that will need changing, I use a frowning face ☹ .

5) Notice that this derivation is *not* the only possible one. As another correct answer, students might see that the $\triangle$ and ∇ terms on the right need to be interchanged, so they might begin this way:

#3
2) $\Box \times (\triangle + \nabla) = (\Box \times \triangle) + (\Box \times \nabla)$
1) $\Box \times (\triangle + \nabla) = (\Box \times \nabla) + (\Box \times \triangle)$

(That is, they began by "flipping" on the "+" sign:

$$\Box \times (\triangle + \nabla) = (\Box \times \triangle) \underset{\uparrow}{+} (\Box \times \nabla)$$
)

7) Yes. cf. answer to question 6.
8) By using Identity 1) .
9) By using Identity 1) . cf. answer to question 6.
10) By using Identity 3) . cf. answer to question 6.
11) Yes.
12) Wayne is right. cf. answer to question 6.
13) Nancy is right.
14) Jerry is right. cf. answer to question 6.
15) He used it correctly. Since $\Box + \triangle = \triangle + \Box$ is an

identity, we can substitute any number into the $\square$, and any number into the $\triangle$.

That is, Identity 3) tells us that, in any sum, we may reverse the order of the two terms. For example:

$$3+5=5+3$$

$$2\tfrac{1}{3}+\tfrac{7}{9}=\tfrac{7}{9}+2\tfrac{1}{3}$$

$$1066+1958=1958+1066$$

$$(a+b)+(c+d)=(c+d)+(a+b)$$

$$(\square+\triangle)+(\triangle\times\triangle)=(\triangle\times\triangle)+(\square+\triangle)$$

$$J_0(\mu x)+J_1(\mu x)=J_1(\mu x)+J_0(\mu x)$$

etc.

List #III

16) Yes. Identities 1), 2), and 3)   Identity 4) . We shall see how in the next question.

17) Yes.

Notes: 1) It is always legal to begin a derivation with a "trivial" identity, such as:

$$(a+b)\times(c+d)=(a+b)\times(c+d)$$

2) To get from i) to ii) in Charlie's derivation, we substitute into Identity #2) :

$$\square\times(\triangle+\bigtriangledown)=(\square\times\triangle)+(\square\times\bigtriangledown)$$

Put $a+b\rightarrow\square$

Put $c\rightarrow\triangle$

Put $d\rightarrow\bigtriangledown$

The result is:

$$\boxed{a+b}\times(\triangle\!\!\!\!\!{\scriptstyle c}\;+\bigtriangledown\!\!\!\!\!{\scriptstyle d}\;)=(\boxed{a+b}\times\triangle\!\!\!\!\!{\scriptstyle c}\;)+(\boxed{a+b}\times\bigtriangledown\!\!\!\!\!{\scriptstyle d}\;)$$

or,

$$(a+b)\times(c+d)=[(a+b)\times c]+[(a+b)\times d]$$

3) The next step is a tactical one. We want to break up

$$[(a+b)\times c]$$

by using Identity 2) again. In order to use Identity 2), we must have the common factor on the left. We now move it there.

$$[(a+b)\times c] = [c\times(a+b)]$$

This is legal because of Identity 1).

4) We have now gotten this far:

iii) $(a+b)\times(c+d) = [c\times(a+b)] + [(a+b)\times d]$

5) We now do the same thing with the term

$$[(a+b)\times d]$$

This gets us to

iv) $(a+b)\times(c+d) = [c\times(a+b)] + [d\times(a+b)]$

6) We now use Identity 2) to break up the term

$$[c\times(a+b)]$$

as follows:

Substitute

$$c \longrightarrow \square$$
$$a \longrightarrow \triangle$$
$$b \longrightarrow \triangledown$$

(Ask the class to decide how to substitute!)

The result is:

$$c\times(a+b) = (c\times a) + (c\times b).$$

This gets us this far:

v) $(a+b)\times(c+d) = (c\times a) + (c\times b) + [d\times(a+b)]$

7) This is a good time to look ahead and see where we are going. We are here:

$$(a+b)\times(c+d) = (c\times a) + (c\times b) + [d\times(a+b)]$$

and we want to get here:

$$(a+b)\times(c+d) = (a\times c) + (a\times d) + (b\times c) + (b\times d)$$

The left hand side is entirely satisfactory:

$$(a+b)\times(c+d) = (c\times a) + (c\times b) + [d\times(a+b)]$$

The first two terms on the right are close to what we want:

$$(a+b)\times(c+d) = (c\times a) + (c\times b) + [d\times(a+b)]$$

The term $[d\times(a+b)]$ is not yet satisfactory.

$$\underset{☺}{(a+b)\times(c+d)}=\underset{☺}{(c\times a)}+\underset{☺}{(c\times b)}+\underset{☹}{[d\times(a+b)]}$$

8) We now break up the term $[d\times(a+b)]$ by substituting into Identity 2). Ask the class to decide how to substitute. The answer is:

$d\longrightarrow\square \qquad a\longrightarrow\triangle \qquad b\longrightarrow\nabla$

We must also "flip" the factors $\underline{a}$ and $\underline{c}$ in the term $(c\times a)$, and so on, until finally we get:

ix) $(a+b)\times(c+d)=(a\times c)+(b\times c)+(a\times d)+(d\times b)$

We must now reverse the $\underline{d}$ and $\underline{b}$ in the last product on the right, and we must use Identity 3) to rearrange the order of addition:

$$(a+b)\times(c+d)=(a\times c)+(a\times d)\times(b\times c)+(b\times d)$$

18) Yes. cf. answer to question #17.

19) cf. answer to question #17.

The justifications, marked in, appear as follows:

i) $(a+b)\times(c+d)=(a+b)\times(c+d)$ It is always legal to start with a trivial identity.

#2 ii) $(a+b)\times(c+d)=[(a+b)\times c]+[(a+b)\times d]$

#1 iii) $(a+b)\times(c+d)=[c\times(a+b)]+[(a+b)\times d]$

#1 iv) $(a+b)\times(c+d)=[c\times(a+b)]+[d\times(a+b)]$

#2 v) $(a+b)\times(c+d)=(c\times a)+(c\times b)+[d\times(a+b)]$

#1 vi) $(a+b)\times(c+d)=(a\times c)+(c\times b)+[d\times(a+b)]$

#1 vii) $(a+b)\times(c+d)=(a\times c)+(b\times c)+[d\times(a+b)]$

#2 viii) $(a+b)\times(c+d)=(a\times c)+(b\times c)+(d\times a)+(d\times b)$

#1 ix) $(a+b)\times(c+d)=(a\times c)+(b\times c)+(a\times d)+(d\times b)$

#3 x) $(a+b)\times(c+d)=(a\times c)+(a\times d)+(b\times c)+(d\times b)$

#1 xi) $(a+b)\times(c+d)=(a\times c)+(a\times d)+(b\times c)+(b\times d)$

20) Since the distributive law (or Identity #2 on our present list) reads $\square\times(\triangle+\nabla)=(\square\times\triangle)+(\square\times\nabla)$, when Larry substitutes, he gets

$$\boxed{a+b}\times(\overset{c}{\triangle}+\overset{d}{\nabla})=(\boxed{a+b}\times\overset{c}{\triangle})+(\boxed{a+b}\times\overset{d}{\nabla})$$

or, in other words,

$$(a+b)\times(c+d)=[(a+b)\times c]+[(a+b)\times d],$$

21) Yes, he does.

List #IV

22) Yes. 1), 2), and 3) imply 4).

23) Marie is right. We shall work out the derivation in answer to question 24.

24) After the preceding problem (List #III), my sixth-grade students find this one easy. (Brian did it immediately in his head, while Alan ostentatiously yawned!) Here is one possible answer:

#1 i) $a + (b \times c) = a + (b \times c)$ ("trivial" identity)

ii) $a + (b \times c) = a + (c \times b)$

#3 iii) $a + (b \times c) = (c \times b) + a$

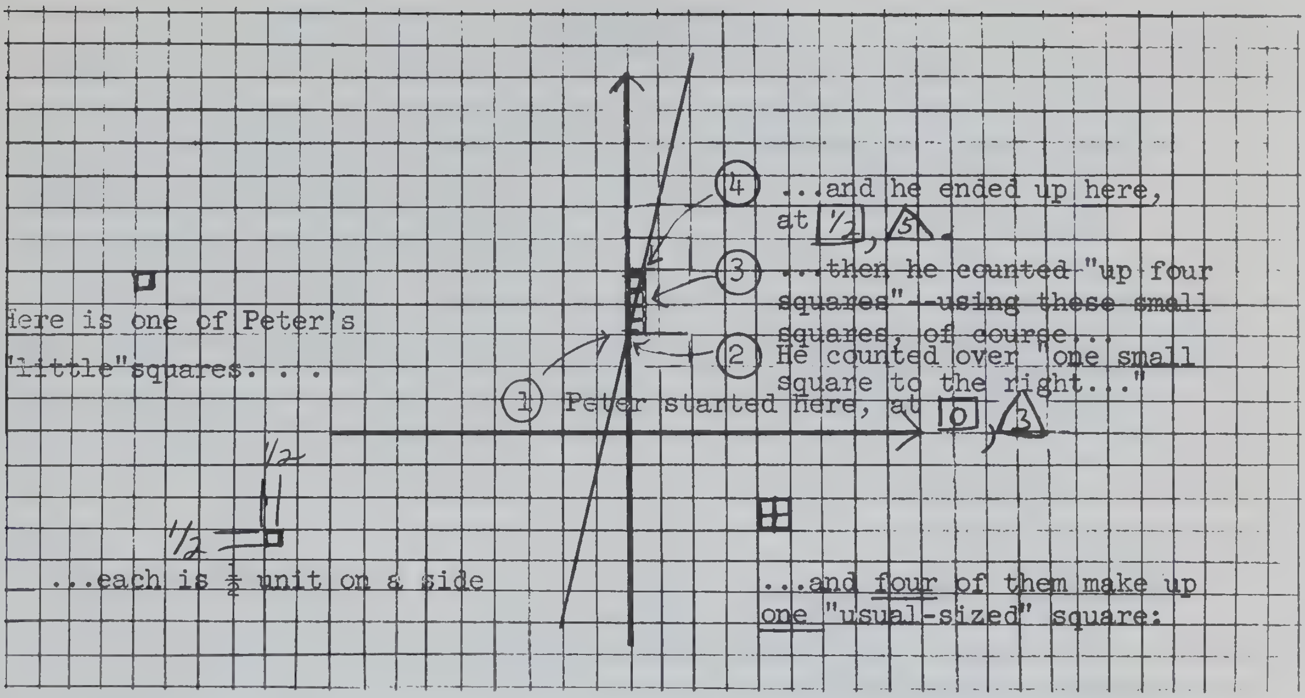
Note: As a matter of fact, Identity 2 was not necessary. Actually,

Identities 1) and 3) $\Longrightarrow$ 4).

You may prefer to present List #IV this way:

List #IV-A

1) $\square \times \triangle = \triangle \times \square$
2) $\square + \triangle = \triangle + \square$
3) $a + (b \times c) = (c \times b) + a$


④ ...and he ended up here, at $\boxed{\tfrac{1}{2}}$, $\triangle\!\!\!\!\!\!{\scriptstyle 5}\;$.
③ ...then he counted "up four squares"--using these small squares, of course...
② He counted over "one small square to the right..."
① Peter started here, at $\boxed{0}$, $\triangle\!\!\!\!\!\!{\scriptstyle 3}\;$
Here is one of Peter's "little" squares. . .
1/2
1/2
...each is ½ unit on a side
...and four of them make up one "usual-sized" square:

16) We arrived at the point $\boxed{\tfrac{1}{2}}$ $\triangle\!\!\!\!\!\!{\scriptstyle 5}\;$ (compare the preceding answer): This is a point of the truth set for the open sentence $\triangle = (4 \times \square) + 3$, as we can see by substituting:

$$\triangle = (4 \times \square) + 3$$

$$5 \searrow \qquad\qquad\qquad \swarrow \tfrac{1}{2}$$

$$\triangle = (4 \times \square) + 3$$

$$\triangle\!\!\!\!\!\!{\scriptstyle 5}\; = (4 \times \boxed{\tfrac{1}{2}}) + 3$$

$$5 = (4 \times \tfrac{1}{2}) + 3$$

$5 = 2 + 3$, which is true.

When we substitute $\boxed{\tfrac{1}{2}}$, $\triangle\!\!\!\!\!\!{\scriptstyle 5}\;$ into the open sentence $\triangle = (4 \times \square) + 3$, we get a true statement. Hence, $\boxed{\tfrac{1}{2}}$, $\triangle\!\!\!\!\!\!{\scriptstyle 5}\;$ does belong to the truth set for the open sentence $\triangle = (4 \times \square) + 3$,

17) A good description might be something like this:

"It doesn't matter what size squares you use. The pattern of a straight-line graph still works just the same way. You can use little squares or big squares."

18) Yes, it does.

CHAPTER 19 - GENERALIZATIONS

I didn't have to write this chapter--a fourth grade class "wrote" it for me. All I had to do was to write it up after the lesson was over.

You may not want to use *all* of these chapters on identities. You may also prefer to change the order.

In our view, the most important thing is to suit the lesson to the class. If your group takes off in full pursuit of the idea of *generalization*, like my fourth-graders did, then you may wish to use Chapter 19 *before* you use Chapter 15 or Chapter 16.

ANSWERS:

1) We can write $\underline{\quad\quad} \times \square = \square \times \underline{\quad\quad}$, and we can put *any* number in the "___", *provided we put this same number in the other "___"*. But this is just what we mean by our usual use of *placeholder notation*. Hence, we can write,

$$\triangle \times \square = \square \times \triangle$$

and if we *add* *this one identity* to Ruth's list, we can delete all of the special cases

$$3 \times \square = \square \times 3$$
$$4 \times \square = \square \times 4$$
$$1960 \times \square = \square \times 1960$$

because they don't tell us anything *new* that we don't already know from the single identity

$$\triangle \times \square = \square \times \triangle$$

2) See the answer to 1) above. If we add the identity

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

to Jerry's list, then we can delete all of the special cases such as

$$\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$$

etc.

3) If the students don't get the answer to question 1), then we suggest that you *don't tell them* or *rush them*. We simply go on to question 3), and hope that, by answering question 3), they will discover how to answer question 1).

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4) If the class discussion comes along well, this question may be superfluous. (David, of course, is right.)

5) Same comment as in 4) above.

6) Cf. answers 1-5, above.

7) Cf. answers 1-5, above.

11) You can insert any number here

$$(\square + \underset{\uparrow}{\underline{\quad\;}}) \times (\square + \underset{\uparrow}{\underline{\quad\;}})$$

provided you also insert twice this number here

$$= (\square \times \square) + (\underset{\uparrow}{\underline{\underline{\quad\;}}} \times \square) + \ldots$$

and the product of this number, times itself, here

$$= (\square \times \square) + (\underline{\underline{\quad\;}} \times \square) + {}'\;\underset{\uparrow}{'}\;{}'$$

Using placeholder notation in the usual way, we can write

$$(\square + \bigtriangledown) \times (\square \times \bigtriangledown) = (\square \times \square) + [(2 \times \bigtriangledown) \times \square] + (\bigtriangledown \times \bigtriangledown)$$

or, if you prefer,

$$(\square + \bigtriangledown) \times (\square \times \bigtriangledown) = (\square \times \square) + [(\bigtriangledown + \bigtriangledown) \times \square] + (\bigtriangledown \times \bigtriangledown)$$

If we add either of these last two identities to Lucille's list, then we can delete all the special cases, corresponding to inserting 3, 5, or 4 into the $\bigtriangledown$.

CHAPTER 20 - DERIVATION OF IDENTITIES BY "REPEATED APPLICATIONS"

This chapter is similar to Chapter 19.

It can be used immediately after Chapter 14, if desired. In fact, for fourth grade groups, we are inclined to recommend this order.

For many groups you may prefer to omit this chapter.

ANSWERS:

1) By repeated applications of $\triangle + \square = \square + \triangle$, we can derive each of the identities listed. For example, here is the derivation of the last one.

We begin with a "trivial" identity:

$$\triangle + 5 + \square + 9 + \bigtriangledown = \triangle + 5 + \square + 9 + \bigtriangledown$$

We usually write this on the blackboard with the starting line at the top, and the "goal" at the bottom, like this:

$$\triangle + 5 + \square + 9 + \bigtriangledown = \triangle + 5 + \square + 9 + \bigtriangledown$$

$$\triangle + 5 + \square + 9 + \bigtriangledown = \bigtriangledown + 9 + \square + 5 + \triangle$$

This greatly helps the students to see where they are going. Moreover, it usually looks familiar, sonce many students have seen word problems such as this:

WELL

TALK

where the object is to fill in the middle lines with words, allowing only one letter change from each line to the next. This "word" problem could be solved as follows:

WELL
TELL
TALL
TALK.

In the same way, to make up a derivation, we want to get from the top line to the bottom line by making one single change at each step:

$$\triangle + 5 + \square + 9 + \bigtriangledown = \triangle + 5 + \square + 9 + \bigtriangledown$$

$$\triangle + 5 + \square + 9 + \bigtriangledown = \bigtriangledown + 9 + \square + 5 + \triangle$$

Here is a "solution," with the intermediate steps filled in:

$$\triangle + 5 + \square + 9 + \bigtriangledown = \triangle + 5 + \square + 9 + \bigtriangledown$$
$$\triangle + 5 + \square + 9 + \bigtriangledown = 5 + \triangle + \square + 9 + \bigtriangledown$$
$$\triangle + 5 + \square + 9 + \bigtriangledown = 5 + \square + \triangle + 9 + \bigtriangledown$$
$$\triangle + 5 + \square + 9 + \bigtriangledown = 5 + \square + 9 + \triangle + \bigtriangledown$$
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$$\triangle + 5 + \square + 9 + \bigtriangledown = \bigtriangledown + \square + 9 + 5 + \triangle$$
$$\triangle + 5 + \square + 9 + \bigtriangledown = \bigtriangledown + 9 + \square + 5 + \triangle$$

The point of this derivation is that the identity ,
which justifies each individual step in the derivation above, permits us to deduce the identity

$$\triangle + 5 + \square + 9 + \bigtriangledown = \bigtriangledown + 9 + \square + 5 + \triangle$$

and consequently the identity $\triangle + 5 + \square + 9 + \bigtriangledown = \bigtriangledown + 9 + \square + 5 + \triangle$ may be deleted from Mary's list.

2) Jerry is right. Here is the derivation:

$$\square + 2 + \bigtriangledown + 3 = \square + 2 + \bigtriangledown + 3$$

(We begin with a "trivial" identity.)

$$\square + 2 + \bigtriangledown + 3 = 2 + \square + \bigtriangledown + 3$$

(Here we used the commutative law of addition, $\square + \triangle = \triangle + \square$, to reverse the order of 2 and $\square$.)

$$\square + 2 + \bigtriangledown + 3 = 2 + \bigtriangledown + \square + 3$$

$$\square + 2 + \bigtriangledown + 3 = 2 + \bigtriangledown + 3 + \square$$

$$\square + 2 + \bigtriangledown + 3 = 2 + 3 + \bigtriangledown + \square$$

$$\square + 2 + \bigtriangledown + 3 = 3 + 2 + \bigtriangledown + \square$$

$$\square + 2 + \bigtriangledown + 3 = 3 + \bigtriangledown + 2 + \square$$

3) Yes, it is. Of course, there are other ways of filling in the intermediate steps which are also correct.

4) He used the commutative law of addition, to "flip" the $\square$ and 2 on the symbol $\square + 2$, ending up with the result $2 + \square$.

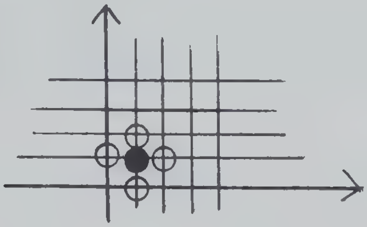
5, 6, 7) cf. answer to question 1.

8) We have had great success with this question. We do not mean to insist on any special answer at this stage; rather, we just want to get the children wondering about this.

LESSON FOLLOWING CHAPTEP 20

You may want to insert a review lesson here.

As a second possibility, you may prefer to devote this lesson to playing the point-set game. In order to give this a fresh "new" look, you can change the rules slightly, so as to bring the game closer to the authentic Japanese game of "GO". (In particular, you can stop counting connections along diagonals, and consider only connections along the horizontal and vertical lines. Thus



the ● at (1, 1) is not considered "connected" to the "breathing space" (i. e., unoccupied points) at (0, 0), (2, 0), (0, 2), and (2, 2). Consequently, under these new rules, the ● at (1, 1) is surrounded and must be erased.

(For information about the authentic rules for "GO", you can write to:

The American Go Association
96 Cedar Avenue
Hackensack, New Jersey

or else to

Katagiri Garden City, Inc.
931 Franklin Avenue
Garden City, New York)

We recommend that you approach the "Go" rules with some care. Our modified rules make the game much easier. "Go", played with the authentic rules, is a far more interesting game, but it is very much harder, and there is some danger of confusing the students. In this situation we suggest you use your own discretion.

21. NAMES FOR NUMBERS

The idea of this chapter is to introduce some language that greatly simplifies work with identities. Those of you who have studied logic, or who are familiar with the Beberman-Illinois-UICSM program, will recognize the idea we are using. (An excellent reference is Exner and Rosskopf, Logic in Elementary Mathematics, McGraw-Hill.)

In the first place, we interpret " = " to mean that we are dealing with two names for the same number.

Thus, "$2+2=4$" will be interpreted to mean that "$2+2$" is one name for a certain number, and "4" is another name for that same number.

In an identity, we can proceed as in this example:

$$\square + (3+4) = \square + (4+3)$$

We know these are equal, because "$3+4$" is one name for some number, and "$4+3$" is another name for this same number.

22. THE AXIOMS

This lesson has an important point--but, as usual, it is better not to "drive it home." It is just an idea that we want the students to think about.

I actually taught this lesson to a 5th grade class in Weston, Connecticut. I'll tell you the results as we go through the answers:

ANSWERS:

1) Actually, it ought to look pretty much like the list of "Axioms" given toward the end of this chapter.

My class in Weston brought in this list:

$$\square \times 1 = \square$$

$$\square + 0 = \square$$

$$\square \times 0 = 0$$

$$\square + \triangle = \triangle + \square$$

$$\square - \square = 0$$

$$\square \times \triangle = \triangle \times \square$$

$$\square + \square = 2 \times \square$$

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

$$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + [(\triangle + \triangle) \times \square] + (\triangle \times \triangle)$$

I told them that this was really a pretty good piece of work that they had just done--"after all," I said, "it took mathematicians 200 or 300 years to arrive at a similar list."

"What took them so long?" Mary asked.

"Well, after all," I said, "you know your list is not quite right. One of those identities doesn't belong on the list. Do you know which one it is?"

During the rest of the perios they worked out a correct derivation of the last identity $[(\square \times \triangle) + (\square \times \triangle) = \ldots]$, starting from the "trivial"

identity

I was astonished! This is *real* *mathematics*. . .

(This list of axioms, of course, is the usual list of *axioms for a field*, except that we have thus far avoided any explicit discussion of the associative laws, and we have avoided inverses as much as possible; $\square - \square = 0$ is the only exception. You may be interested in reading *A Concrete Approach to Abstract Algebra*, by W. W. Sawyer, published by Freeman, and available in a paperback edition.)

2) First, we can shorten it by *generalization*: for example, the identities

$$\begin{gathered} 3 \times \square = \square \times 3 \\ 4 \times \square = \square \times 4 \\ 5 \times \square = \square \times 5 \\ \vdots \end{gathered}$$

can all be crossed off and replaced by the single identity:

$$\triangle \times \square = \square \times \triangle$$

Similarly, the identities

$$\begin{gathered} (\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9 \\ (\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9 \\ \vdots \end{gathered}$$

can all be crossed off and replaced by the single identity:

$$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + [(\triangle + \triangle) \times \square] + (\triangle \times \triangle)$$

Besides *generalization*, there are various forms of *implication*. Any statement can be crossed off the list if we can make up a *derivation* for it using only the *other* "axioms", and beginning with a "trivial" statement.

Thus, $(\square \times 3) + (\triangle \times 5) = (5 \times \triangle) + (3 \times \square)$ can be crossed off the list. It is *implied* by the identities $\square + \triangle = \triangle + \square$ and $\square \times \triangle = \triangle \times \square$ as the following derivation shows:

$$\begin{gathered} (\square \times 3) + (\triangle \times 5) = (\square \times 3) + (\triangle \times 5) \\ (\square \times 3) + (\triangle \times 5) = (\triangle \times 5) + (\square \times 3) \\ (\square \times 3) + (\triangle \times 5) = (5 \times \triangle) + (\square \times 3) \\ (\square \times 3) + (\triangle \times 5) = (5 \times \triangle) + (3 \times \square) \end{gathered}$$

q. e. d.

It may be well to illustrate implication also with some verbal examples: for instance, on the list:

Today is June 20.
Today is my birthday.
My birthday is June 20.

A student in
class made this up!!

we can cross off any one of the statements, and the other two will imply it.

3) We answered this under question 1.

Answer to final question:

The list of "axioms" and "names for numbers" is of fundamental importance in mathematics, because from this list we can derive every other identity that exists.

23. SIGNED NUMBERS

This lesson is, in a sense, review.

At this stage, however, my classes are still unreliable in questions of the arithmetic of signed numbers. This lesson is intended to help somewhat--though I wouldn't expect the entire class to learn arithmetic for signed numbers even by the end of this lesson. There will be more chances in the future.

24. USING IDENTITIES

This is an easy lesson, intended to show how identities can be used in "ordinary" arithmetic.

Some teachers prefer to emphasize

$$\square \times \frac{\triangle}{\triangle} = \square$$

as a basis for the quotient of two fractions.

As you become familiar with the algebraic way of thinking, you may like to use identities frequently in your work with arithmetic.

ANSWERS: 1) Yes

2) Yes

3) Yes

4) Elizabeth is right. Jerry used the left hand side, whereas Alan used the right hand side. (Substitute 4 into the $\square$, 2 into the $\triangle$, and $\frac{1}{4}$ into the $\bigtriangledown$. The result is:

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

$$\boxed{4} \times (\overset{2}{\triangle} + \overset{1/4}{\bigtriangledown}) = (\boxed{4} \times \overset{2}{\triangle}) + (\boxed{4} \times \overset{1/4}{\bigtriangledown})$$

$$4 \times (2 + 1/4) = (4 \times 2) + (4 \times \tfrac{1}{4})$$

This side says add before multiplying.	This side says "multiply first, then add."

10) Yes. This is another identity that you can use in studying arithmetic.

25. OPEN SENTENCES AND SIGNED NUMBERS

In this lesson, the emphasis is on discovery.

ANSWERS:

1) $^{+}2 \times {}^{-}3 = {}^{-}6$
2) $^{-}2 \times {}^{-}5 = {}^{+}10$
3) $\{2, 11\}$
4) $\{2, 3\}$
5) $\{7, 2\}$
6) $\{^{+}5, {}^{-}3\}$ Note: It is easier to look first for factors of -15, and then to select the pair which will add to yield $^{+}2$:

$$^{+}5 \times {}^{-}3 = {}^{-}15$$
$$^{+}5 + {}^{-}3 = {}^{+}2$$

7) The factors of $^{-}10$ include: $^{+}1, {}^{-}1, {}^{+}2, {}^{-}2, {}^{+}5, {}^{-}5, {}^{+}10, {}^{-}10$

Which pair will add to give us $^{+}3$?

ans.: $^{+}5 \times {}^{-}2 = {}^{-}10$ $^{+}5 + {}^{-}2 = {}^{+}3$

Hence, the truth set is $\{^{+}5, {}^{-}2\}$

8) $\{^{+}7, {}^{-}2\}$
9) $\{5, 10\}$
10) $\{^{-}3, {}^{-}2\}$ since $^{-}3 \times {}^{-}2 = 6$

$$^{-}3 + {}^{-}2 = {}^{-}5.$$

CHAPTER 26 - GRAPHS

This is a short chapter (it usually takes us about 15 minutes, but it sometimes takes longer), and one which our students usually enjoy.

The chapter is in two parts.

In part I (Questions 1-12) the idea is this:

To mark the truth set of the equation $\triangle = (2 \times \square) + 1$ on a graph, we must begin by locating one point by means of computing:

$$\triangle = (2 \times \square) + 1$$

← put some number here--say, perhaps, 2

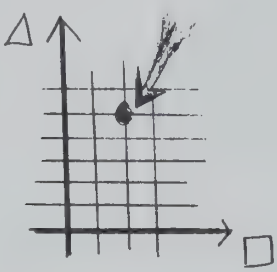
$$\triangle = (2 \times \boxed{2}) + 1$$

Then we compute $[2 \times 2 = 4,\ 4 + 1 = 5]$, and find

$$\triangle\!\!\!\!\!\:{\scriptstyle 5} = (2 \times \boxed{2}) + 1$$

that we need a "5" in the $\triangle$, in order to make a true statement.

We would mark $\boxed{2}$, $\triangle\!\!\!\!\!\:{\scriptstyle 5}$ on a graph here:



Now we come to the important part!!

To find another point, we could also compute:

For example, put 3 in the $\square$, and compute:

$$\triangle = (2 \times \boxed{3}) + 1$$

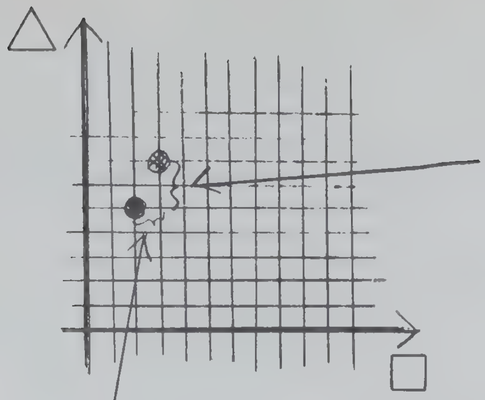
← We need a "7" in the $\triangle$.

However--IT IS NOT NECESSARY TO COMPUTE AFTER YOU HAVE FOUND SOME FIRST POINT.

Instead of "computing", we can use the secret pattern:

$$\triangle = (2 \times \square) + 1$$

← Since this number is a "2", we can start from our first point, count over 1 square to the right, and count 2 squares up:

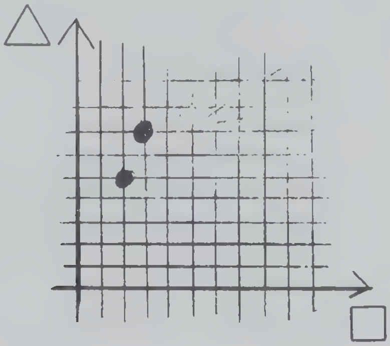


and then count up 2 squares:

$\triangle = (2 \times \square) + 1$

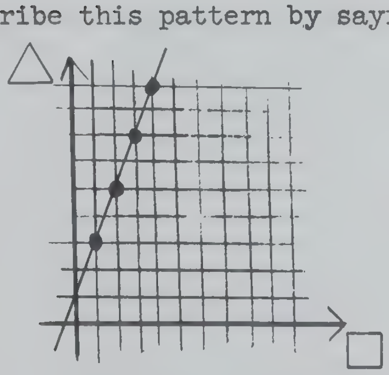
Count one square to the right: ● ⟶

In this way you locate a second point:



Remarks (Mathematical):

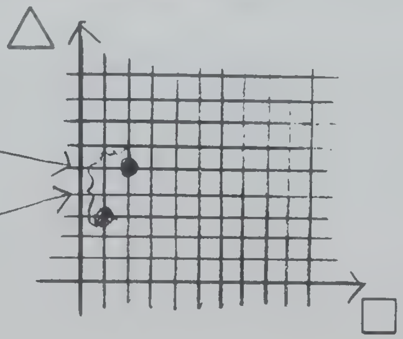
1) We could also describe this pattern by saying that the points lie on a straight line:



2) Of course, the pattern works also to the left, as well as to the right, but when we count one square to the left, we then count two squares down:

Count one square to the left ←●

and then 2 squares down



3) The number here: $\triangle = (\underset{\uparrow}{2} \times \square) + 1$

determines the "secret pattern".

For example, in this equation $\triangle = (\underset{\uparrow}{7} \times \square) + 2$,

the pattern would be: "1 square to the right, and seven squares up".

4) The number here: $\triangle = (7 \times \square) + \underset{\uparrow}{2}$

does not have any effect on the "secret pattern".

5) Of course, this "secret pattern" is studied in algebra and calculus; the technical name for it is "slope".

6) An equation must be of this form:

$$\triangle = (\underset{\underset{\text{Some number here}}{\uparrow}}{\_\_\_\_} \times \square) + \underset{\underset{\text{Some number (not necessarily the same) here.}}{\uparrow}}{\_\_\_\_\_\_}$$

in order to have the "secret pattern" work.

The secret pattern will work for these equations

$$\triangle = (5 \times \square) + 3$$

$$\triangle = (7 \times \square) - 2$$

$$\triangle = (2 \times \square) + 5$$

$$\triangle = (3 \times \square) - 3$$

$$\triangle = \underset{\uparrow}{\square} + 1$$

Notice that this is the same as:

$$\triangle = (1 \times \square) + 1$$

The "secret pattern" will not work for these equations (at least, not unless you exercize some ingenuity and change things around a bit):

$$\triangle + \square = 25$$

$$\square = (3 \times \triangle) + 2$$

$$2 \times \triangle = 3 \times \square$$

$$\triangle = \square \times \square$$

(This is explained fully, under the heading "slope of linear equations," in almost any good text on algebra or calculus.)

Pedagogical Remarks:

1) How are students supposed to learn this "secret pattern"?

We do *not* tell them!

They discover it from experience with graphs. Until they discover this pattern, they must *compute* each point *by substituting into the open sentence*; *after* they discover this "secret pattern", they need to substitute and compute *for only one point*--they can locate all the other points by using the "secret pattern."

2) You may have to keep pushing--some students are content to substitute and compute. You have to keep asking them if they have discovered a "secret pattern", so that they can find points *without* substituting and computing.

Questions 13-18 form Part II of this chapter.

Here, we make another effort at getting students to recognize this "secret pattern" and to use it. We *give* them a graph (i.e., a picture of the *truth set*), and ask *them* to write the equation. In general, they cannot do this without using the idea of the slope.

ANSWERS:

1) Yes

2) Yes

3) Yes. "over one (to the right), and up two."

4) No. The point *should* be one line above where he put it. Don't *tell* the students that, though! It's *their* job to discover the pattern.

5) No. Nancy's point is "over one (to the right), and up one" from the preceding point. This is *not* the correct pattern. She should have put the point "over one (to the right) and up two."

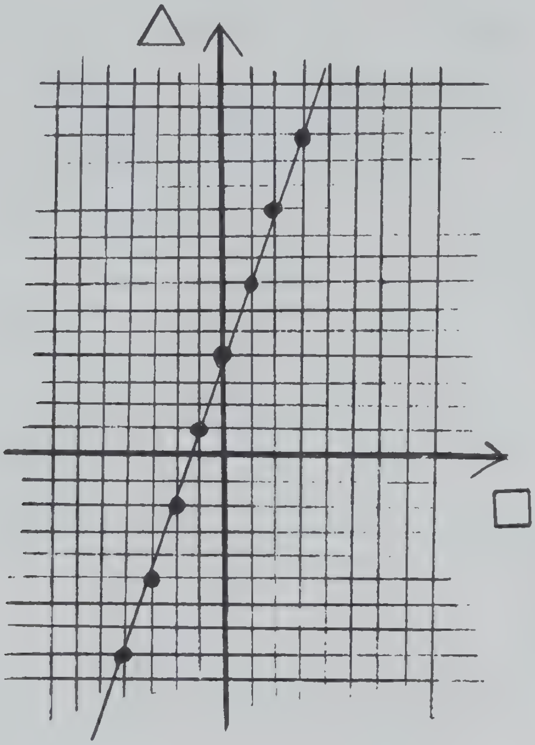
6) Yes. (Or, at least, his point is in the right place!)

7) Yes.

8) Yes.

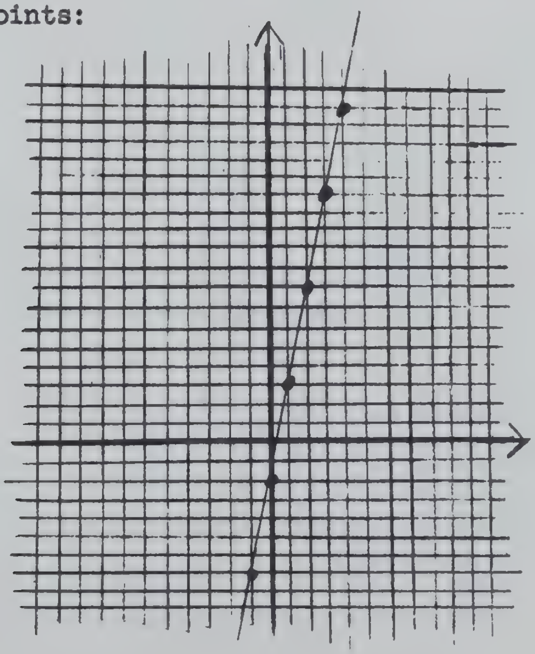
9) Yes

10) The correct pattern is "over one (to the right) and up three"--or, equivalently, "over one to the left, and down three." Here are some correct points:



11) Yes.

12) The correct pattern is "over one (to the right) and up five." Here are some correct points:



13) We get the answer by several steps:

1st, the pattern is evidently "over one (to the right) and up two". Therefore, the equation must be

$$\triangle = (2 \times \square) \pm \underline{\ ?\ },$$

where we do not yet know whether to use the + or — , and we do not yet know what number toes in the spot marked $\underline{\ ?\ }$.

Now: guess some number for the $\underline{\ ?\ }$ spot. Suppose we try 2.

Then the equation would be $\triangle = (2 \times \square) + 2$

But the points from this equation fall below those on the graph.

Consequently, we increase our guess, and try

$$\triangle = (2 \times \square) + 3$$

This works!

14) $\triangle = (3 \times \square) + 1$

15) $\triangle = (5 \times \square) + 2$

16) $\triangle = \square + 3$

17) $\triangle = (7 \times \square) - 6$

18) $\triangle = (2 \times \square) + 2$

27. EQUATIONS

This is a good chance to check up on your pace.

We feel that a too-slow pace defeats itself. As I have said earlier, we prefer the "light touch" and the *brisk pace*.

This really does work, and I hope you will try it.

This chapter--Section 27 on "Equivalent Equations" and "Transform Operations"--is, for us, *one single 45 minute lesson.*

That really is all the time it takes.

(I might add that this lesson usually works very nicely. I have taught it with gratifying results to fourth graders, fifth graders, sixth graders, and eighth graders--in every case, allowing a single 45 or 50 minute period.

Of course, like anything else in teaching, it isn't guaranteed. The one time it really fell flat was with a sixth grade class, just before Christmas, when a representative of the *Wall Street Journal* was observing the lesson!

That, I suppose, is life. . .)

Outline: We do three things in this lesson:

i) We use two observations to permit us to solve equations such as $3+(11\times\square)=29$.

ii) We study *transform operations*.

iii) We use *transform operations* to "solve" linear equations.

This will be explained in detail in the next few pages.

Answers:

Questions 1-18 correspond to the *first* objective listed above.

1. $\{2\}$ (This is just to remind the class of what an equation is like.)

2. It must be a fraction, because we find (by substituting) that 2 is too small:

$$3+(11\times 2)$$
$$3+22$$
$$25 \text{ (less than 28)}$$

3 is too large:

Hence, the answer must be a fraction between 2 and 3.

3. Of course, what we have written in answer to Question 2 does prove it.

Pedagogical suggestion:

We ask this question ("Can you prove it?"). While the students are eagerly trying to explain a proof, we turn half-way to the blackboard and begin to write:

too small	too large

At this stage we get clear statements from the class that 2 is too small, so we add it to our list on the blackboard:

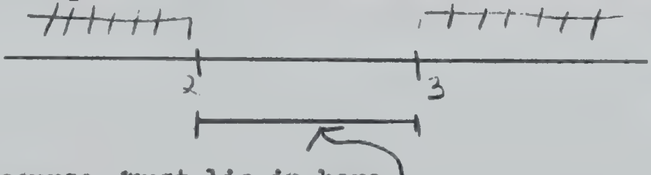
too small	too large
2	

When someone in the class insistently informs us that 3 is too large, we enter it on our blackboard list.

too small	too large
2	3

With this much help, the class can explain why the answer must be a fraction--namely, because it's between 2 and 3.

4. The point of this question is to introduce the so-called "number-line picture":



The answer, of course, must lie in here.

5. The point of this question is to goad the children into observing that the denominator of the fraction must be 11. That is, the only possibilities are 2 1/11, 2 2/11, 2 3/11, 2 4/11, 2 5/11, 2 6/11, 2 7/11, 2 8/11, 2 9/11, 2 10/11. Of course, only one of these actually works, and in a moment we shall

find which one that is.

The reason that $2\frac{1}{2}$ cannot work is that 11 x $2\frac{1}{2}$ will be a fraction.

Hence $3 + (11 \times 2\frac{1}{2})$ will be a fraction.

Hence $3 + (11 \times \boxed{2\frac{1}{2}})$ cannot equal 28.

6. (11 x $\square$) must be a whole number,

since

$$3 + \boxed{(11 \times \square)} = 28,$$

If we add a fraction to 3, the result will also be a fraction, so it cannot equal the whole number 28.

7. (Cf. the answers to questions 5 and 6.) The denominator must be 11. (That is, 11 x 2 5/11, for example, will be a whole number, whereas--for example,

11 x 2 1/3

would not be a whole number.)

8. Marie, of course, is right.

9. The point of this question is to extend the "number-line picture" so that it shows that the only possible answers are 2 1/11, 2 2/11, 2 3/11, 2 4/11, 2 5/11, 2 6/11, 2 7/11, 2 8/11, 2 9/11, and 2 10/11. We still don't know which one of these actually works.

We do not want to be in a hurry, here, to get the answer! It is this method of analyzing the problem that is the important thing . . . !

10. We substitute in order to decide:

$$3 + (11 \times \square) = 28$$

$$3 + (11 \times 2\tfrac{9}{11}) = 28$$

$$3 + (22 + 9) = 28$$

(Note that we used the distributive law

$$\square \times (\nabla + \Delta) = (\square \times \nabla) + (\square \times \Delta)$$

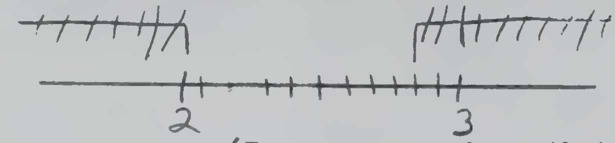
in order to multiply quickly.)

$$3 + 31 = 28$$

$$34 = 28$$

False--and we see that 2 9/11 is too large!

11. Bill's picture should now look like this:

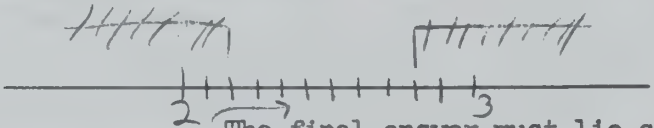


(I. e., we now know that 2 9/11 is too large.)

12. Several possible anssers. In any event, the interval of "prospective candidates" will be narrowed by shrinking from the left.

13. Similar to 11 and 12.

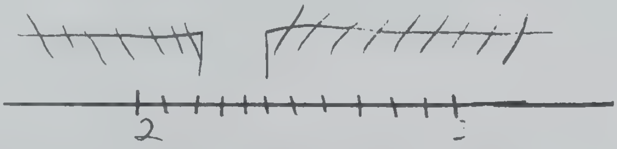
13. Suppose, for example, that a student in your class found that 2 2/11 is toc small. The picture would then look like this:



The final answer must lie somewhere in here.

If a student tries 2 5/11, he will find that it is too large.

Bill's picture should now be extended to look like this



This picture shows that the actual answer must either be 2 3/11 or 2 4/11. (2 1/11, and 2 2/11 have been excluded as "too small," and 2 5/11, 2 6/11, etc., have been excluded because they are "too large.")

The actual answer is 2 3/11, as we can find by substituting:

$$3+(11\times\square)=28$$

$$3+\left(11\times\boxed{2\tfrac{3}{11}}\right)=28$$

$$3+(22+3)=28$$

$$3+25=28$$ True!!

So the truth set is: $\left\{2\tfrac{3}{11}\right\}$

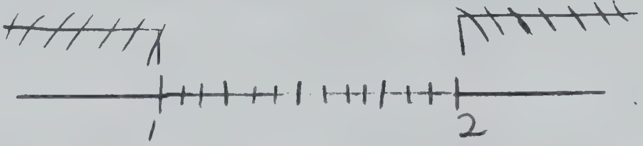
Questions 15-18. For each of these, we use the same procedure as we did with $3+(11\times\square)=28$,

For example, in problem 15, we first ask the class:

"Without finding the answer, can you tell me anything about it?"

Yes, it must be a fraction,
with denominator 13, between 1 and 2.

We now draw this picture on the board:



We now try a few guesses (hoping that we won't hit the answer too soon, since it is the process that we want to study).

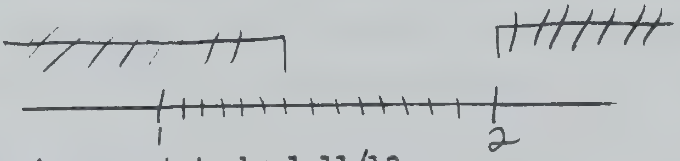
Suppose a student guesses 1 6/13. (This is a good 1st guess, since it divides the interval approximately in half. We usually compare this to the questions asked by the panel on "What's My Line?". A question like "Do you work indoors rather than outdoors?" gives far more information that a question like "Do you rustle long-horn steers on a medium-sized ranch in Utah?" The answer to this second question is very likely to be "No," and you have learned very little.)

By substituting, we find:

$$5+(13\times\square) = 29$$
$$5+(13\times\boxed{1\tfrac{6}{13}}) = 29$$
$$5+(13+6) = 29$$
$$5+19 = 29$$
$$24 = 29$$

False, and we see that 1 6/13
is too small!

The picture now looks like this:



The final answer turns out to be 1 11/13.

This completes the first part of the day's lesson.

Questions 19-53 form the second part of the lesson. Here the objective is to get the children to find specific examples of "equivalent equations", and to *generalize* from these specific examples and hence to arrive at the 5 basic transform operations, which are:

1) You may add the same number to each side of an equation.
2) You may subtract the same number from each side of an equation.
3) You may use an identity (for example, the identity $\triangle + \square = \square + \triangle$).
4) You may multiply both sides of an equation by the same number.
5) You may divide both sides of an equation by the same number.

We do not tell these laws to the students--we try to get the children to tell *us* the laws.

Incidentally, an important further purpose of this section is to give the class *some* understanding of the right way to *use* these laws. For example, if we go from

$$5 + (2 \times \square) = 11$$

to

$$4 + (3 \times \square) = 12 ,$$

we have *not* proceeded correctly, and the two equations shown are *not* "equivalent".

Something to beware: some of the children may try to use the word "equivalent" in its ordinary *everyday sense*. This is never permissible in mathematics. "Equivalent," as we use it here, is a *technical term*, and we say that two equations are "equivalent" *if they have the same truth set*--this is the *only* meaning that is allowed. (If you have studied "work" in physics, or "energy" in chemistry, or the "gainfully employed" in economics, or "radicals" in mathematics, you are familiar with this problem. The *technical* meanings of these words *are not the same* as their *everyday* meanings.)

Answers:

19. Yes, because each has the truth set $\{3\}$. Since their truth sets are the same, they are equivalent.

20. Yes, since each has the truth set $\{4\}$. (The point of this

question, or course, is to suggest to students the possibility of adding the same number to each side of the equation. We hope the suggestion if not too blatantly obvious.)

You may prefer to omit questions 21-53, and to replace them with a "continuing list" of your own on the blackboard.

Personally, I recommend that you use your own "continuing list", and that you name the methods after the students in your own class who first use them successfully.

28. "THINK OF A NUMBER"

The object of this lesson is to provide experience in using placeholders. All of our subsequent work with word problems will depend mainly upon the children feeling natural and at ease in using the placeholder to "hold a place" for some unknown number.

At the start of this lesson (which takes us about 50 minutes), we allocate one minute or less to questions 1 and 2, in order to provide the children with a gentle reminder that we have studied products of signed numbers.

Answers:

1) $^-2 \times {}^+3 = {}^-6$

2) $^-2 \times {}^-5 = {}^+10$

(Personally, we run through questions 1-7 briskly, so that children are not tempted to linger. We don't encourage them to ask us questions: we ask the questions, they figure out the answers. Of course, we try to keep the tone of our class warm, friendly, encouraging, and moderately permissive--but, at this point, we are not eager to discuss the arithmetic of signed numbers, or the solution of quadratic equations--we are eager to "get on with the game. . ." We want the children also, to be eager to "get on with the game. . ." This is fun!!

When our students have been interviewed, they have indicated that they enjoy the brisk pace of our classes, and the feeling that you have to stay on your toes all the time. . .)

Questions 3-7 are, similarly, a gentle reminder that we have studied equations of this type.

At this point--for the first time in the course--we mention in a casual and matter-of-fact way (we are all co-conspirators in having discovered this a long time ago in the distant past) that the product of the numbers in the truth set must be this number

$$(\square \times \square) - (5 \times \square) + \underset{\uparrow}{6} = 0$$

and the sum of the numbers in the truth set must be this number

$$(\square \times \square) - (\underset{\uparrow}{5} \times \square) + 6 = 0$$

The truth set, therefore, must be $\{2, 3\}$. You may wish to substitute 2 in (mainly so that the children will remember that substituting is the fundamental "last-word" test).

Answers:

3) $\{2, 3\}$ $\quad 2 \times 3 = 6$

$2 + 3 = 5$

4) $\{2, 11\}$ $\quad 2 \times 11 = 22$

$2 + 11 = 13$

5) $\{12, 8\}$ $\quad 12 \times 8 = 96$

$12 + 8 = 20$

(Note that it is easier to consider factors of 96 first, and then try to pick out two factors that will add to give 20.)

6) The factors of ${}^{-}15$ include ${}^{+}1, {}^{-}1, {}^{+}3, {}^{-}3, {}^{+}5, {}^{-}5, {}^{+}15, {}^{-}15$.

Which pair will add to give us ${}^{+}2$? Evidently,

$${}^{+}5 \times {}^{-}3 = {}^{-}15$$
$${}^{+}5 + {}^{-}3 = {}^{+}2$$

Hence, the truth set is $\{{}^{+}5, {}^{-}3\}$

7) The factors of ${}^{-}6$ include: ${}^{+}1, {}^{-}1, {}^{+}2, {}^{-}2, {}^{+}3, {}^{-}3, {}^{+}6, {}^{-}6$.

Which pair will add to give ${}^{-}1$? Evidently,

$${}^{-}3 \times {}^{+}2 = {}^{-}6$$
$${}^{-}3 + {}^{+}2 = {}^{-}1$$

Hence, the truth set is $\{{}^{-}3, 2\}$

Now we start the game!

I suggest you do not refer to questions 8-15 during the class period. They are included to give you some idea of how the game goes, and to give the children some reference material that they can read after class, in case they forget how the game worked, and are curious about it.

Personally, I would not assign this material.

This is just a game that we play during class.

This is just fun!

In teaching this lesson to fourth or fifth graders, I proceed this way:

I say:	I write on the board:
Jimmy, think of a number. . .	#
. . .Now multiply it by itself. . .	# x #
. . . Now subtract twice the original number [this may call for some discussion. . .]	- 2 x #
. . . Now add 5 . . .	+5
. . .All right, now tell me the final result. . .	

Jimmy says "13" I write 13

The blackboard now looks like this:

$$\begin{array}{c} \# \\ \# \text{ x } \# \\ -2 \text{ x } \# \\ +5 \\ \hline 13 \end{array}$$

Now we come to the next stage!

But, before we look at this "next stage", let me show you what Jimmy has presumable written on his paper. (Naturally, I have not seen Jimmy's paper--nor have the other students.)

When I said	Jimmy *presumably* wrote:
Jimmy, think of a number. . .	4
. . . Now multiply it by itself. . .	4 x 4 = 16
. . . Now subtract twice the original number [if necessary explain this more fully]	2 x 4 = 8 16 - 8 = 8
. . . Now add 5 . . .	8 + 5 = 13

Now let's look at the next step in the class activity--we coax one of the students into writing a suitable open sentence.

Me: "O. K. -- Who wants to write the equation. . .?"

Somebody usually eager to come to the board and try it -- even though he probably doesn't know yet just what he's supposed to do -- but he wants to get in on the "game". . .

Me: [I intend to try to help steer him in the right direction.] "Let's see -- you don't know the number that Jimmy thought of, so you can represent it by a □ . . ."

Perhaps the student can now start writing:

$$(\square\times\square)-(2\times\square)+5=13.$$

This, of course, is the right answer.

We now come to the third stage -- rewriting the equation in "standard form"!

Our method for solving quadratic equations won't work on the equation

$$(\square\times\square)-(2\times\square)+5=13,$$

until we get a zero on the right hand side:

________________________ = 0.

How can we do this?

That's easy, subtract 13 from each side of the equation. I do that like this:

I say:	I write:
All right, we can subtract 13 from each side of the equation. What will we have on the right?	
[Student says "0".]	= 0
. . . and on the left we have. . .	$(\square\times\square)-(2\times\square)$ . . . = 0

I say:	I write:
. . . now, let's see, we made $5, and we lost $13. . . [Student says " ⁻8 ".]	$(\square \times \square) - (2 \times \square) + {}^{-}5 = 0$

Now--the fourth stage: we find the truth set:

The factors of ⁻8 include: ${}^{+}1, {}^{-}1, {}^{+}2, {}^{-}2, {}^{+}4, {}^{-}4, {}^{+}8, {}^{-}8$.
Which pair will add to give us ⁺2?

That's easy: $${}^{+}4 \times {}^{-}2 = 8$$ $${}^{+}4 + {}^{-}2 = {}^{+}2$$

So the truth set is $\{{}^{+}4, {}^{-}2\}$

At this point, I usually say this to my fourth or fifth graders:
Me: "Well, that's as far as we can go with mathematics. Now we have to use psychology to guess whether Jimmy though of ${}^{+}4$ or ${}^{-}2$. Which do you think Jimmy used, ${}^{+}4$, or ${}^{-}2$?

Class: "Positive four"

Me: "Jimmy, what number did you use?"

Jimmy: "Positive four."

I enjoy this lesson -- so does the class . . .
I hope you will also!!

NOTE ON AVOIDING EXPOSITION

In our classes we usually make a great effort (unnoticed, we hope, by the students) at avoiding exposition. It is all too easy to fall into the "trap" of "explaining something," or telling students.

There are two ways one can fall into this trap:

The students can commit a series of errors that entice you into "setting them straight," or a student can ask a direct question.

We feel this matter is important enough that we have inserted this note, based on observations of our own classes. You may not wish to follow our procedure, but, in case you do, we want to explain it as clearly as possible.

First, there are two likely misunderstandings of the Madison Project procedure.

1) We use far less exposition -- and far more student discovery -- than most teachers are willing to believe is possible. In most non-Project classes that we have observed, there is a very large amount of direct exposition; the teacher will tell the students some rule about how to use parentheses, or how to add signed numbers, or how to solve linear equations, etc. Nearly all of this sort of exposition is eliminated from most Project classes.

Now -- as Max Beberman has remarked -- what do we lose if we do not tell these rules to the students? Virtually nothing! The rules are so complicated when expressed in verbal terms, that any student who can understand and use the rules, could equally well add signed numbers without the rules.

Perhaps the real point is that verbal rules are unnatural in mathematics -- those who are adept at mathematics operate, not by following memorized rules, but rather by relying on a non-verbal familiarity and understanding. How do you go from the dining room to the kitchen in your own home? You do not follow a memorized rule: "1) Stand up. 2) Face door to foyer. 3) Walk to door. 4) Enter foyer. 5) Walk to kitchen door. 6) Enter kitchen door. 7) Once in kitchen, procede as appropriate."

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I doubt that there is a single reader of these notes who would use this method of going from one room to another in his own home. (Possibly a blind person might do so, and this simile is very revealing: if you lack familiarity, you may be forced to rely on rules. But if you are familiar with the matter in question, you would never once be tempted to resort to the procedure of following verbal rules.)

What is even more important, there is probably not a single reader of these notes who would ever use this method of teaching someone else to go from one room to another (unless, again, we were teaching a blind person).

What would we do? We'd say: "Oh, by the way, the kitchen is over here."

That is to say, we'd show them.

I'd like to relate an incident from one of my own classes:

I was trying to get the class to discover the "legal" transform operations on linear equations. So far so good -- they knew clearly which operations were "legal" and which were not. They called them "Brian's method," "Paul's method," etc, after the students in the class who introduced each method.

(For example, going from

$$8 + \square = 23$$

to the equivalent equation

$$9 + \square = 24$$

was "Paul's method," and going from

$$8 + (4 \times \square) = 48$$

to the "equivalent" equation

$$4 + (2 \times \square) = 24$$

was "Brian's method".)

Now -- at this stage I wanted to get them to describe these rules verbally. As a pedagogical device in stating the question, I said:

Me: "Now, suppose someone came into the class here, and you wanted to explain to him what these transform operations are, what would you say?"

I was hoping for answers like: "Well, you can add the same number to both sides of an equation." I did finally get these answers.

But the first answer I got (from an attractive, blonde girl named Susan) was both unexpected and very gratifying:

Susan: "Why, I'd show him."

Secretly, I agreed this would be much the wisest thing to do. In fact, I told Susan I thought it"was a good idea, but what I wanted was some way to tell this newcomer what the legal transform operations were."

The students, who already knew what the transform operations were (because they had seen them and used them), were now able to describe them correctly in words.

But how far would you get describing a cat to someone who had never seen a cat?

The first misconception about Project classes, then, is to assume that we really "explain things pretty much the usual way." We do not. In fact, we do not explain things at all. Well -- as Gilbert and Sullivan put it -- "hardly ever . . ."

2) . . . and that brings us to the second misconception. Project classes are not 100% "non-directive" or lacking in leadership.

We always try to do what is most appropriate at that moment, and this sometimes -- but rarely! -- involves "telling the students" something or other.

We do sometimes "tell them" -- but very rarely, and when we do, the way in which we do it is probably very important. We do not ever "tell" them anything dogmatically, or authoritatively, nor do we emphasize what we are "telling" them.

Rather, when we do take the lead in verbalizing something, we say it casually, as something of an aside, in a tone that implies this is something familiar that we have all known for a long time . . .

. . . sort of a man-to-man statement of a trite remark that we make (with apologies for its triteness) only to insure effective communication . . .

In a sense, we "plant" the remark in the discussion. Later on, when a student finally achieves understanding, we hope he will think he discovered the idea for himself.

The second misunderstanding, then, would be to think that we never direct the students, and never supply a verbalization. We do -- but casually, and without emphasis!

(Students have remarked that one reason they enjoy Project classes is that you must always stay on your toes; you cannot miss something and expect that the teacher will always repeat it, anyhow. Actually, they are wrong -- we repeat things frequently, but it is gratifying to find that the students think that we are always going on to something new and exciting.)

So much for the misconceptions.

Now -- how do we avoid exposition?

Well, we have said earlier that there are two traps. Let's consider the first, where student errors make you want to tell them what they ought to be doing:

An incident from a Project demonstration class at Chappaqua, New York, helps to illustrate how one can sometimes avoid exposition.

Prior to any systematic discussions of signed numbers, we were working with quadratic equations, such as

$$(\square \times \square) - (8 \times \square) + 15 = 0.$$

In the course of substituting 4 into this "open sentence," we encountered this:

$(4 \times 4) - (8 \times 4) + 15 = 0$ True, or false?

$16 - 32 + 15 = 0$ True, or false?

We wished to focus attention on the sum and product rules for quadratic equations, so we did not want to make a production out of the problem of adding

$$16 - 32 + 15 = ?$$

Consequently, we suggested -- in an off-hand way -- that a good method would be to imagine that we are operating a store.

Now, did

16 ______________________________

mean a profit of $16, or a loss of $16?

"A loss of $16," the class assured us.

[This was a demonstration class, with the Superintendent, several principals, and a number of Chappaqua's excellent teachers observing. We were taken somewhat aback!]

But if you really want the kids to discover things, you've got to give them leeway to make mistakes, without hovering over them like an authoritarian oracle . . .

. . . so we went on:

"Now, how about this?

_______________ - 32 _______________

Is this a loss of $32, or a gain of $32?"

Class: "A gain of $32."

[Well, at least they were consistent. In fact, their logic here was impeccable, even if their convention was somewhat unorthodox!]

Teacher: "Now, how about this"

_____________________ 15

Is this a loss of $15, or a gain of $15?

Class: [Still being consistent!] "A loss of $15."

Here was the trap!

Would we respond to the temptation to tell them -- however gently -- that what they were doing was all right, but that people usually used a convention that "+" meant "a gain," and "- " meant a loss?

We made a last ditch fight to avoid exposition.

Teacher: "Now, let's see. Suppose I have 10 - 8. Would that be $10 made and $8 lost, or $10 lost and $8 made?

Class: "Oh, that would be $10 made and $8 lost."

HURRAY!! Our last-ditch battle ended victoriously. At this point the class went back and reversed their interpretation of

$$16 - 32 + 15 = ?\ ,$$

using now the standard convention that we make $16 and $15, for total gain of $31, and we lost $32, so that was a poor day for business. When all was said and done, we were $1 poorer than when we opened shop on that morning.

Would it have done any harm to tell the class? No, of course not; but this was one tree in the forest. We want the over-all class atmosphere to be one where the teacher asks, and the students think and answer and have fun doing it. If you want an oak forest, you must plant oak trees.

If we want the over-all class atmosphere to be one where the teacher asks, and the students answer, then we had best make this the form of each little item that comes up -- at least, in so far as seems wise.

This brings us to the second trap. Sometimes the teacher is tempted into exposition because a student asks a question.

We do sometimes answer such questions -- but usually in an apologetic tone that implies that what we are saying is trite and familiar (much as if we had said that "the winter in Syracuse, of course, tends to be cold").

But we do not usually answer such questions!

If a student says he doesn't understand the rules of the matrix game, we usually say, in a reassuring tone of voice, "Well, wait a minute and see if you don't catch on."

We use this kind of answer if we feel that a child needs a little reassurance.

Sometimes, however, we feel that a child wants to "hand us the problem," and stop thinking about it himself. When a child asks this kind of question (say, for example: "Should I add or subtract?") we would respond by saying: "That's your problem." (The class accepts this in a spirit of good humor -- in fact, they find it fun!)

Teachers often ask us: "What do you do when a child asks ________?

We usually tell the teacher (quite truthfully) that that particular question never arises in Project classes.

Why not?

Actually, we wish we knew! However, we conjecture that the answer is that we maintain an atmosphere where students are supposed to think things through for themselves, that we maintain an atmosphere that respects intellectual curiosity and "guessing" and trial-and-error, that there is no unpleasant feeling associated with "a good idea, but it didn't work," and that we always try for careful readiness-building:

by the time a student sees

$$3 + (2 \times \square) = 8,$$

he has already become familiar with

$$3 + \square = 5$$

and

$$7 + (2 \times \square) = 21,$$

and so on.

Speaking personally, I always enjoy working with elementary school children on this material. Their creative discoveries are just as exciting as if we were working on the frontiers of science, looking for a cure for cancer, or trying to send a man to the moon, or something like that.

As far as the children are concerned, we are working on the frontiers of knowledge.

They enjoy it, and so do I.

I hope that you and your classes will, also. . .

Where Do We Go From Here?

This year (1959-60) I am teaching this material to classes in various grades (in fact, grades 3, 4, 5, 6, and 8). We have been doing approximately the same things in each class, except for the third grade class.

At the time of this writing (January 9, 1960), I have felt the need to pause and think about the future. What should we do with each class in the next few months?

For whatever it may be worth to others, I present here my analysis of what I want to do with my classes in the weeks ahead:

1) I would like to get at least some of these classes familiar with the use of letters in formulas, such as

$$D = \frac{1}{C}\left[AB^2 - \frac{CD-1}{H^2-1}\right].$$

2) I want them to become more familiar with the use of "transform operations" in solving equations.

3) I want them to develop more skill in the use of algebraic identities.

4) I want them to get a deeper understanding of the algebraic structure of arithmetic.

5) I want to work on "word problems" leading to equations or inequalities.

6) I want them to become more secure in the arithmetic of signed numbers.

7) I want more practice in graphing the truth sets of open sentences.

8) I want to introduce some work in geometry.

9) If I feel we have made some secure progress in the areas listed above, then I should like to introduce a few assorted mathematical ideas, such as Euler-Venn diagrams, probability, or possibly more work on logic.

29. GRAPHS

In most of my own classes, at this point, I want a "change of pace" -- hence, I switch from algebraic problems like "Derivations" and "Equations", and begin working for the next few weeks with some co-ordinate geometry.

If you do not wish to turn away from equations just yet, then -- instead of using this lesson--after lesson 28 skip to lesson 37 (Finding Truth Sets.) With a few of my classes I prefer to do this.

ANSWERS:

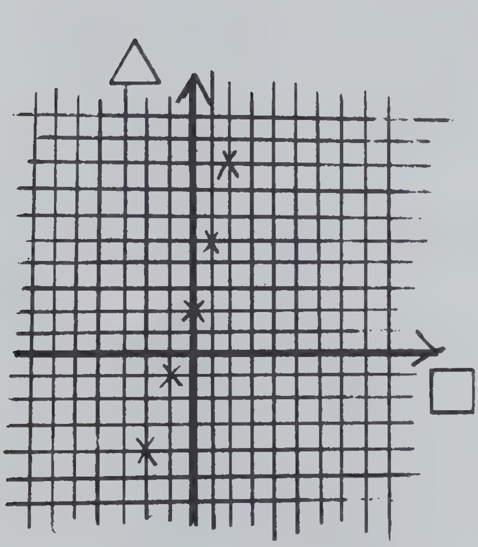
1) For the open sentence $\triangle = (3 \times \square) + 2.$

Here is one answer:

$\square$	$\triangle$
$^{+}1$	$^{+}5$
0	2
-1	$^{-}1$
2	8
3	11
$^{+}4$	$^{+}14$

Table for truth set.

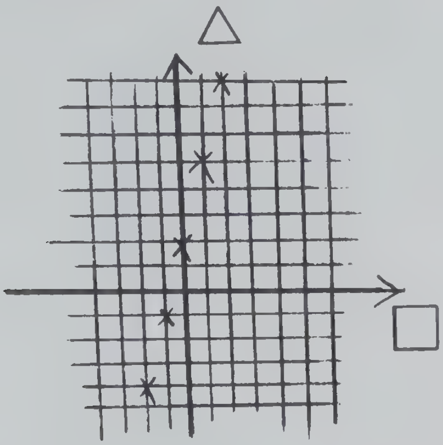
2)



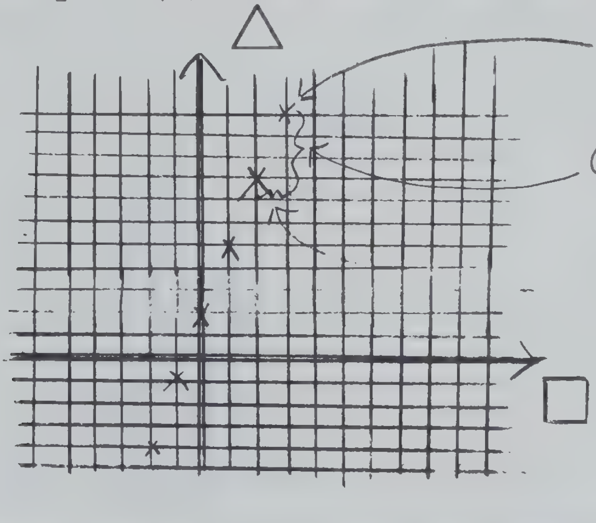
Graph of Truth Set

3) The pattern is "over one to the right, and up three".

4) You can do this, of course, by using the pattern just mentioned. For example, if we have the graph



we can start at our highest point (2, 8) and count "over one, up three," to get:

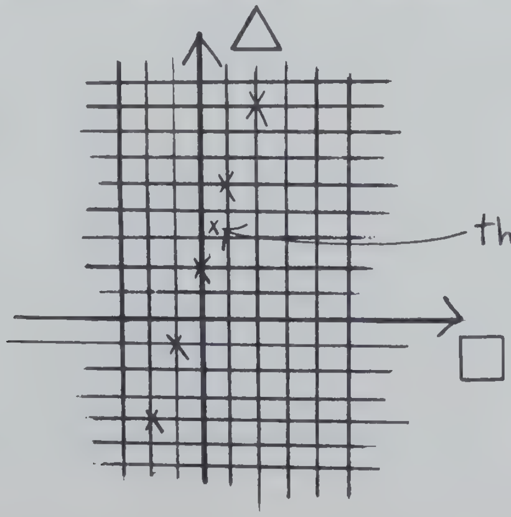


The point goes here, at (3, 11).

② "... and up three"

① "over one to the right..."

5) This can be done by looking at the graph, and observing the pattern of the points. It is clear from looking at this pattern, that the point $\boxed{1/2}$, △? should go here:



the point $\boxed{1/2}$, △?.

Consequently, we see from the graph that the ? in the $\triangle$ is, in fact, $3\frac{1}{2}$. I. e., the point is $(\frac{1}{2}, 3\frac{1}{2})$.

We did not do any arithmetic to find this point. If, however, we now wish to check up on our accuracy by substituting into the open sentence

$$\triangle = (3 \times \square) + 2,$$

we find:

$$\triangle = (3 \times \boxed{1/2}) + 2$$

$$= 3/2 + 2$$

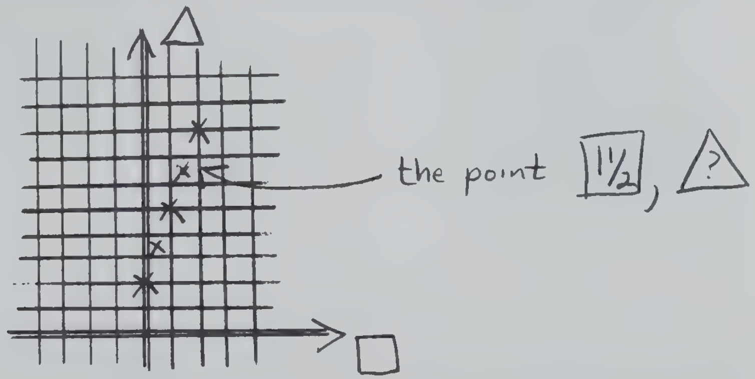
$$= 7/2.$$

In other words, $\overset{\triangle}{3\frac{1}{2}} = (3 \times \boxed{1/2}) + 2$

is a true statement, so $(\frac{1}{2}, 3\frac{1}{2})$ does belong in the truth set.

6) We have just answered this question, in our discussion of question 5.

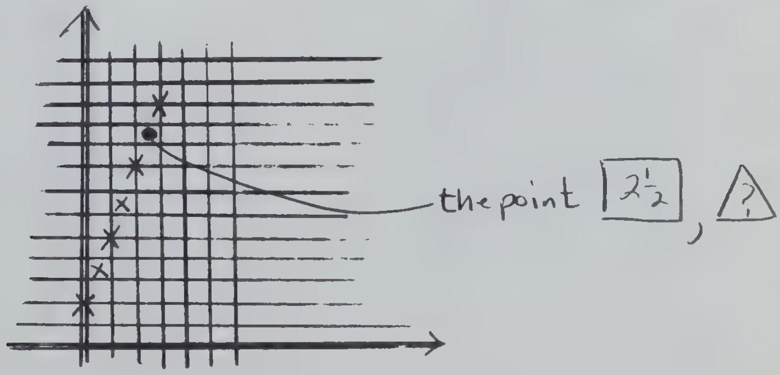
7) By looking at the pattern in the geometry, we see that the point should go here:



We can now read the co-ordinates from the graph; evidently, the point is $\boxed{1\frac{1}{2}}, \overset{\triangle}{6\frac{1}{2}}$

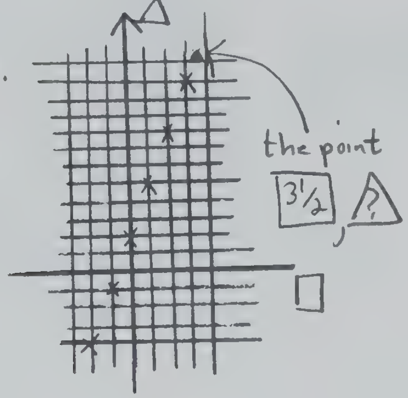
8) $6\frac{1}{2} = (3 \times 1\frac{1}{2}) + 2$ is a true statement.

9) By looking at the pattern of the geometry, we see that the point must belong here:

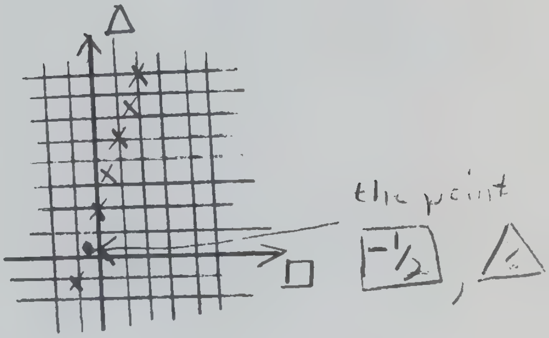


Now that we have marked the point on the graph, we can read off its coordinates, namely $\boxed{2\frac{1}{2}}$, △ $9\frac{1}{2}$.

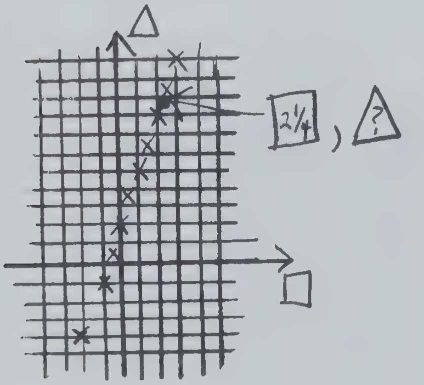
10) a.



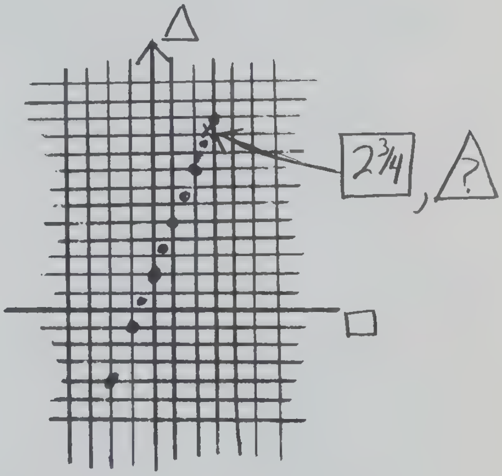
b.



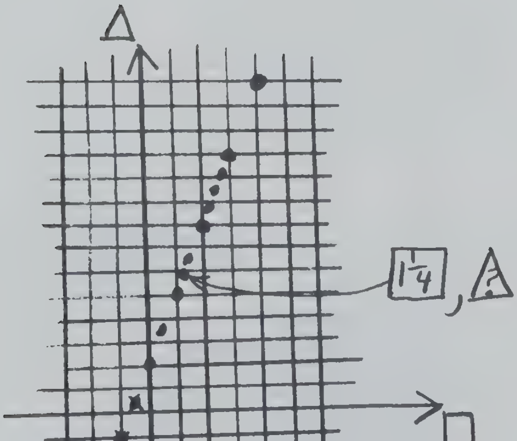
c.



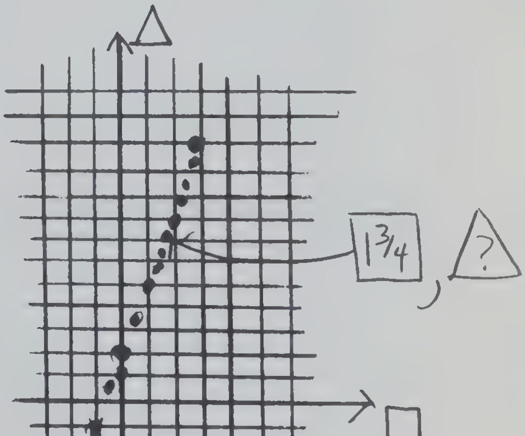
d.



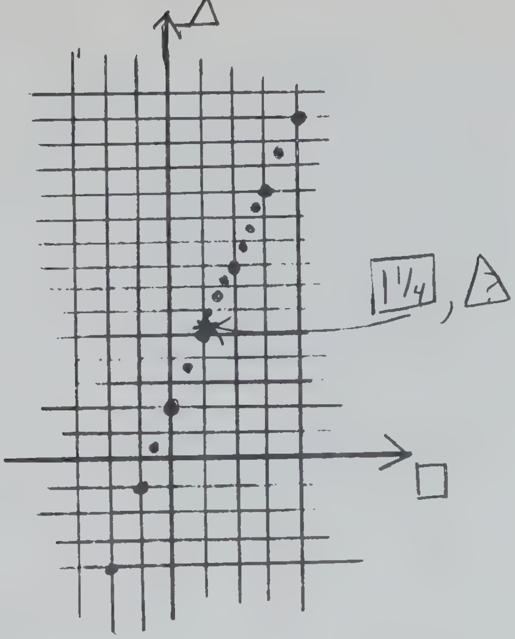
e.



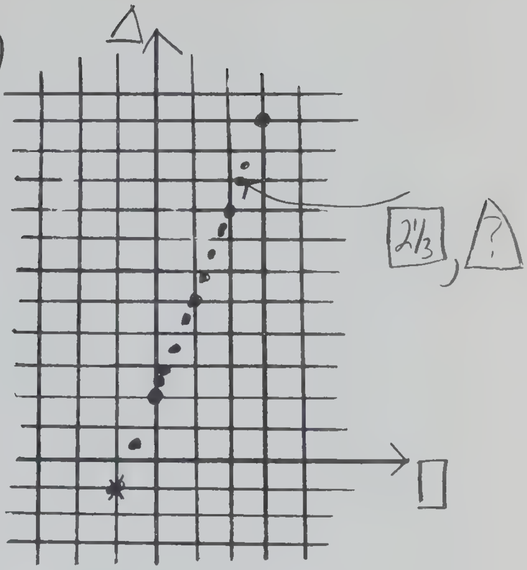
f.



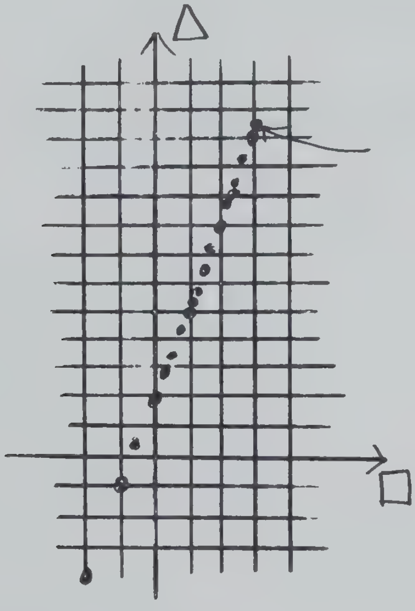
g)



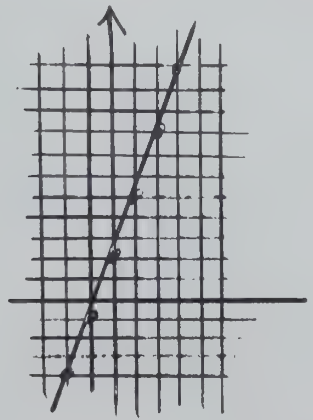
h)



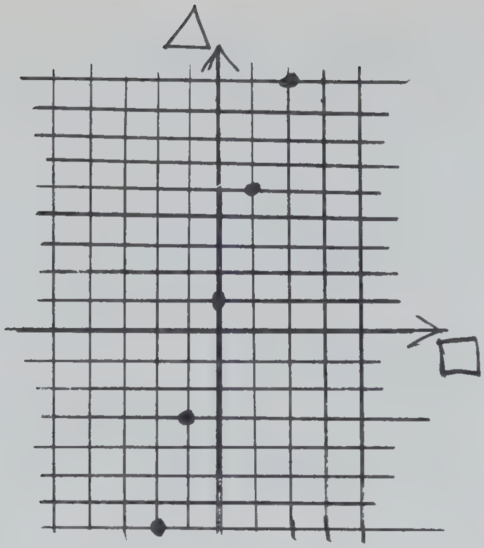
i)



11) As we fill in all these functional points, the graph comes to look like a straight line:



12.)

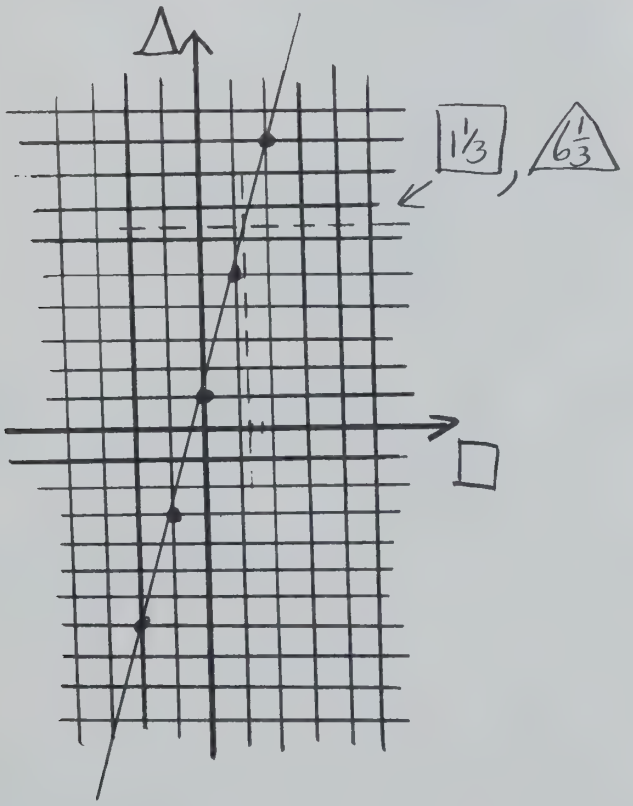


13) The "pattern" is "over one to the right, and up four."

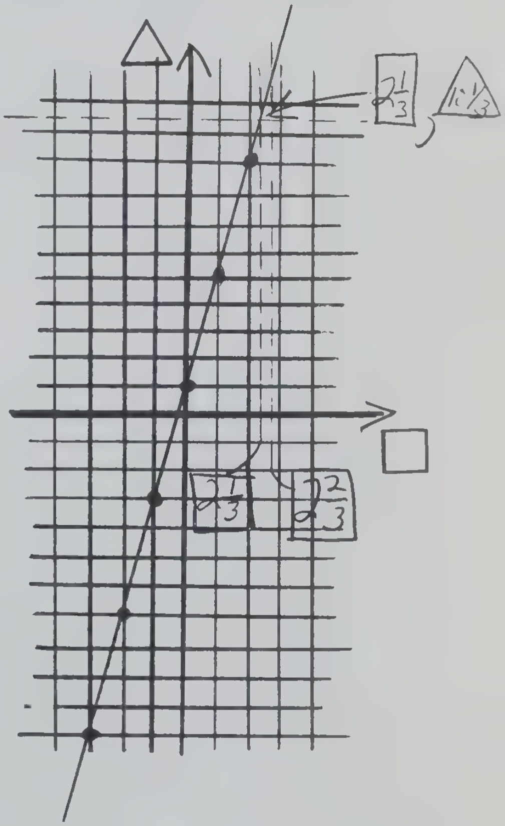
14) No. Jerry is wrong. The pattern is "over one to the right, and up four."

15) and 16) Similar to earlier questions, with $\triangle = (3 \times \square) + 2$

17)



18)



30. A DISCOVERY BY PETER

Answers:

1) The table is correct. (Of course, other number pairs could be used that would also be correct. The only requirement is that the numbers, when substituted into the equation

$$\triangle = (4 \times \square) + 3,$$

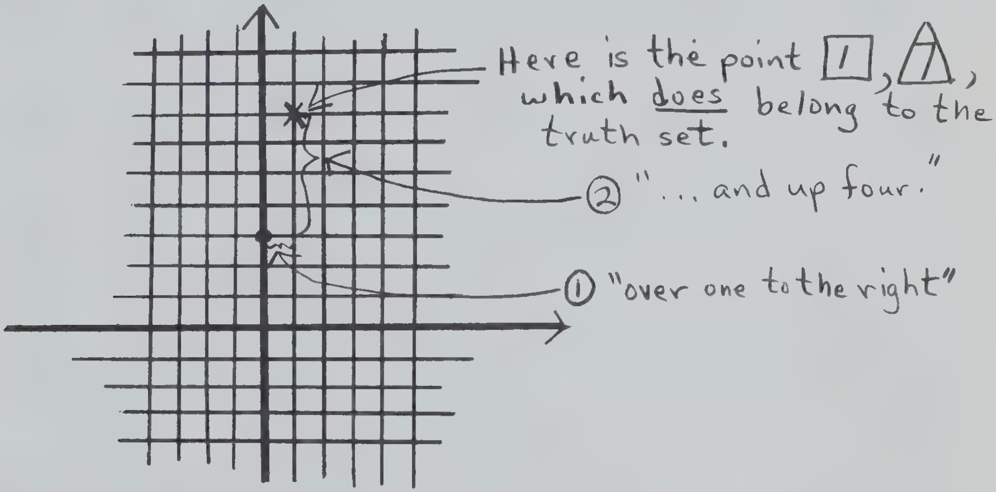
should produce a true statement.)

2) Yes, it is. Of course, many other correct points could be added.

3) Debbie is right.

4) Yes.

5) Yes. If we start at $\boxed{0}\,,\ \triangle\!\!\!\!\!\!{\scriptstyle 3}$ and use Debbie's pattern, we arrive at $\boxed{1}\,,\ \triangle\!\!\!\!\!\!{\scriptstyle 7}$:


Here is the point $\boxed{1}\,,\ \triangle\!\!\!\!\!\!{\scriptstyle 7}$, which does belong to the truth set.
② "... and up four."
① "over one to the right"

6) Yes. You arrive at $\boxed{0}\,,\ \triangle\!\!\!\!\!\!{\scriptstyle 3}$. This point (0, 3) does belong to the truth set, since, on substituting, we get

$$\triangle\!\!\!\!\!\!{\scriptstyle 3} = (4 \times \boxed{0}) + 3,$$

which is a true statement.

7) Yes, since

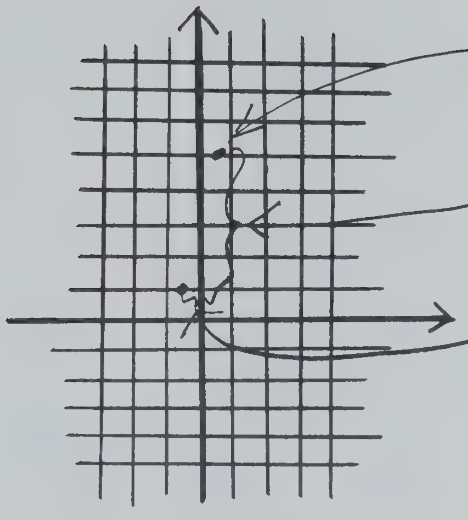
$$1 = (4 \times {}^{-}\tfrac{1}{2}) + 3,$$

or

$$1 = {}^{-}2 + {}^{+}3,$$

is a true statement.

8) Yes



③ . . . and we arrive at $\boxed{\tfrac{1}{2}}$, $\triangle\!5$, which does belong to the truth set. . .

② ". . . and up four. . ."

① "over one to the right. . ."

9) Bill is right.

10) Yes. Substitute into the open sentence. The result will be a true statement.

$$\triangle = (4 \times \square) + 3$$

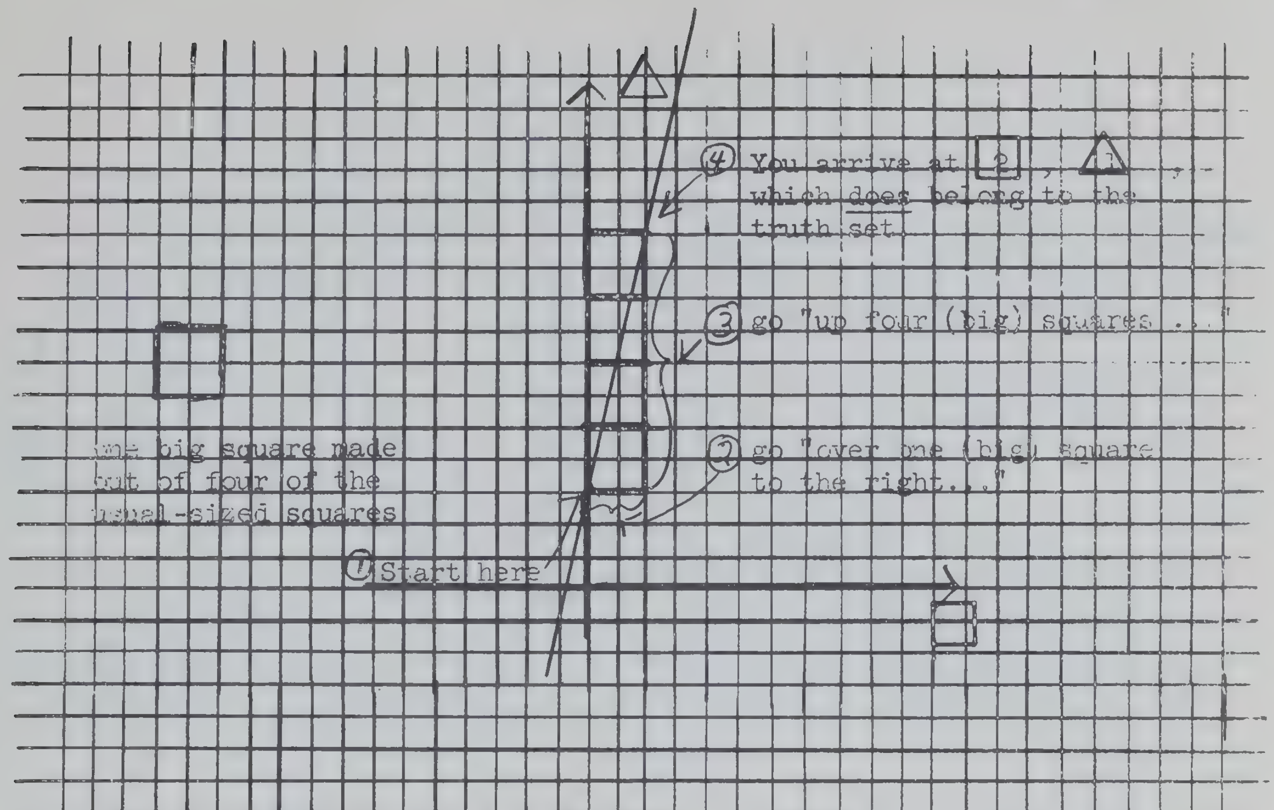
$$\overset{5\,\searrow}{\triangle} = (4 \times \overset{\swarrow\frac{1}{2}}{\square}) + 3$$

$$\triangle\!5 = (4 \times \boxed{\tfrac{1}{2}}) + 3$$

$$5 = (4 \times \tfrac{1}{2}) + 3$$

$5 = 2 + 3$, which is true. Consequently, $\boxed{+\tfrac{1}{2}}$, $\triangle\!5$ does belong to the truth set for the open sentence $\triangle = (4 \times \square) + 3$.

11) Yes. For example, suppose we "put four standard-sized squares together to make one big square." Debbie's pattern would then work like this:



④ You arrive at $\boxed{2}$, △11,
which does belong to the
truth set.

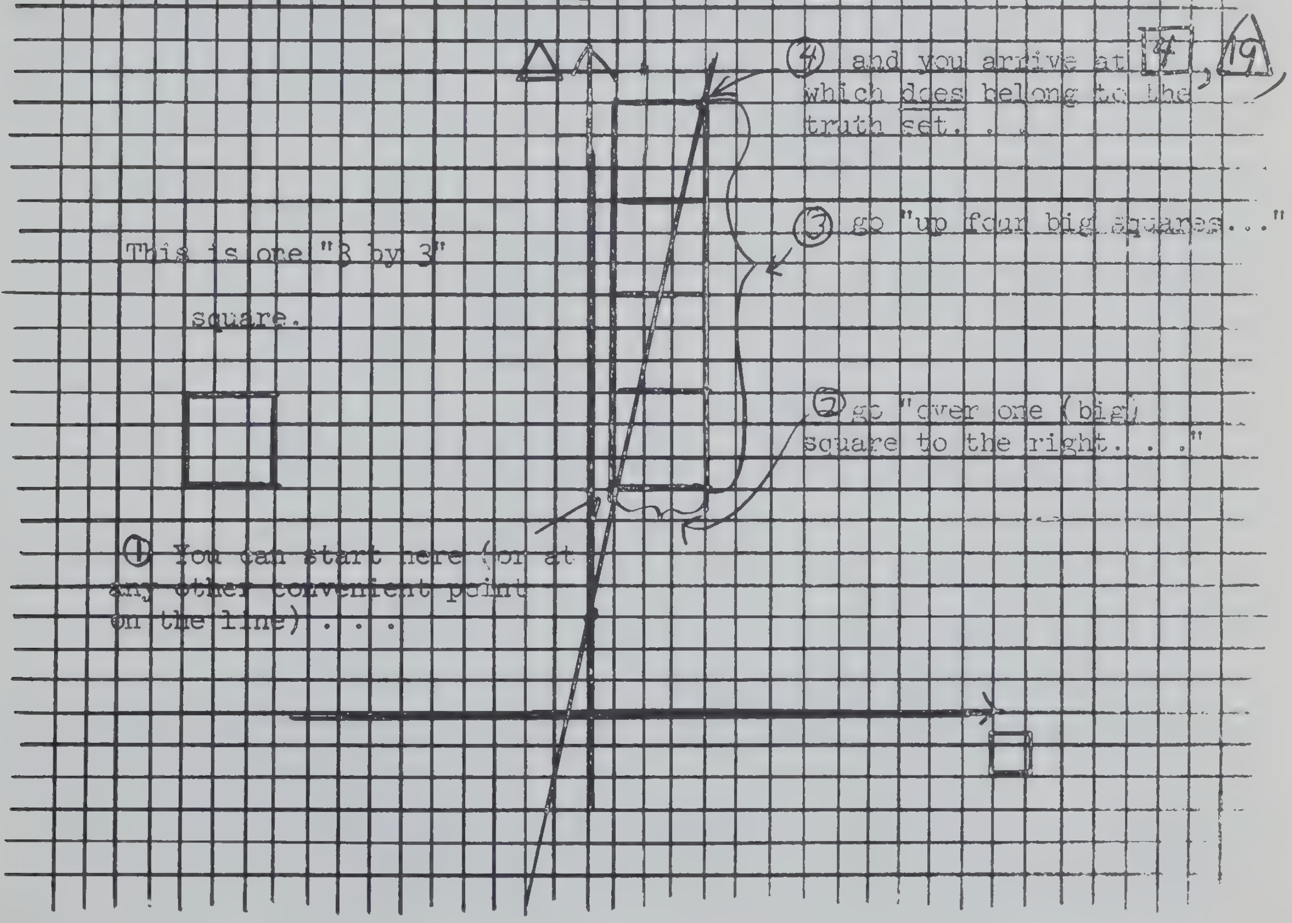
③ go "up four (big) squares . . ."

② go "over one (big) square
to the right. . ."

one big square made
out of four of the
usual-sized squares

① Start here

You could also use "3 x 3" squares, like this:



④ and you arrive at $\boxed{4}$, △19,
which does belong to the
truth set. . .

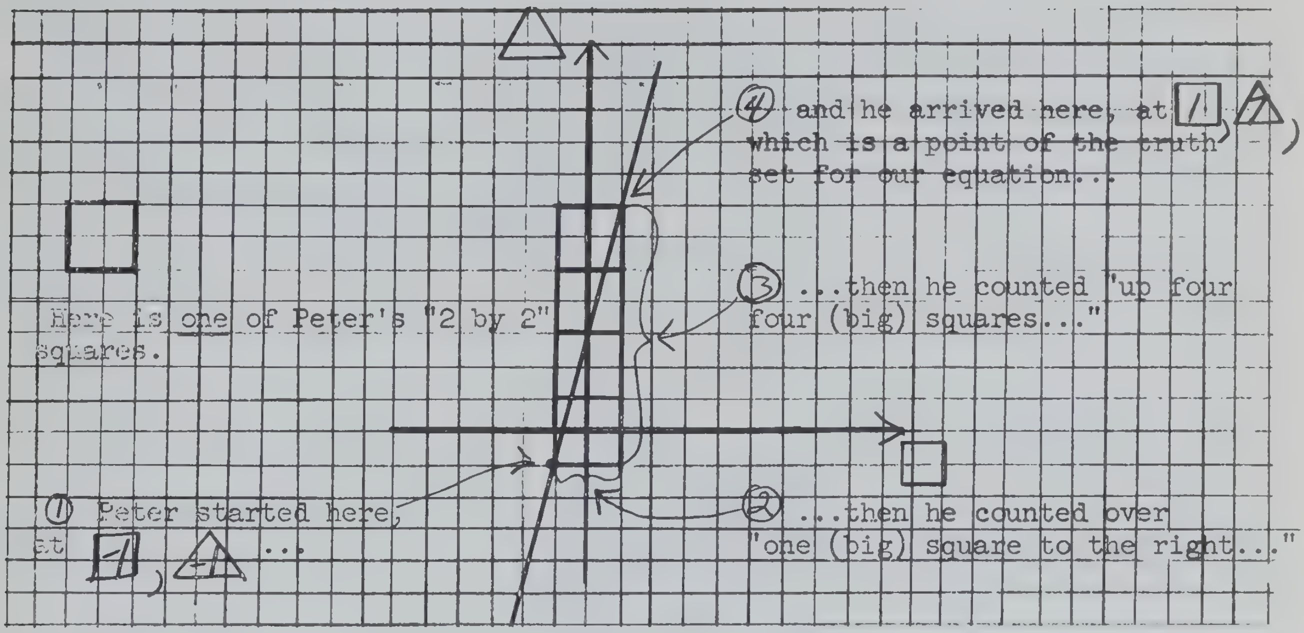
③ go "up four big squares. . ."

This is one "3 by 3"
square.

② go "over one (big)
square to the right. . ."

① You can start here (or at
any other convenient point
on the line) . . .

12) Here is the complete version of Peter's diagram:


④ and he arrived here, at $\boxed{1}, \triangle 7$) which is a point of the truth set for our equation...
Here is one of Peter's "2 by 2" squares.
③ ...then he counted "up four four (big) squares..."
① Peter started here, at $\boxed{{}^{-}1}, \triangle {}^{-}1$ ...
② ...then he counted over "one (big) square to the right..."

13) We ended at $\boxed{1}, \triangle 7$. This is a point of the truth set for the open sentence $\triangle = (4 \times \square) + 3$ How do we know? That's easy! Just substitute into the open sentence. The result will be a true statement:

$$\triangle = (4 \times \square) + 3$$

$${}^{+}7 \searrow \triangle = (4 \times \overset{{}^{+}1 \swarrow}{\square}) + 3$$

$$\triangle{}^{+}7 = (4 \times \boxed{{}^{+}1}) + 3$$

$${}^{+}7 = (4 \times {}^{+}1) + 3$$

$${}^{+}7 = 4 + 3,$$ which is, of course, true. Hence $\boxed{{}^{+}1}, \triangle{}^{+}7$ does belong to the truth set of the open sentence $\triangle = (4 \times \square) + 3$.

14) I guess I'll leave it up to you to decide whether the children's descriptions are reasonably good or not.

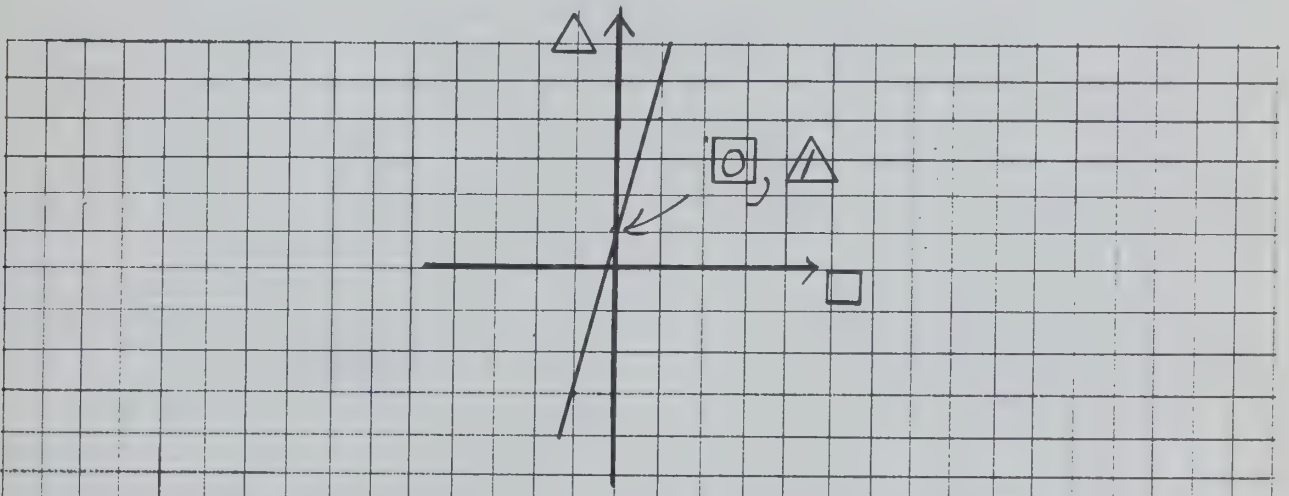
15) Here the complete version of Peter's picture:

31. PRECISION DRAFTING, AND READING NUMBERS FROM GRAPHS

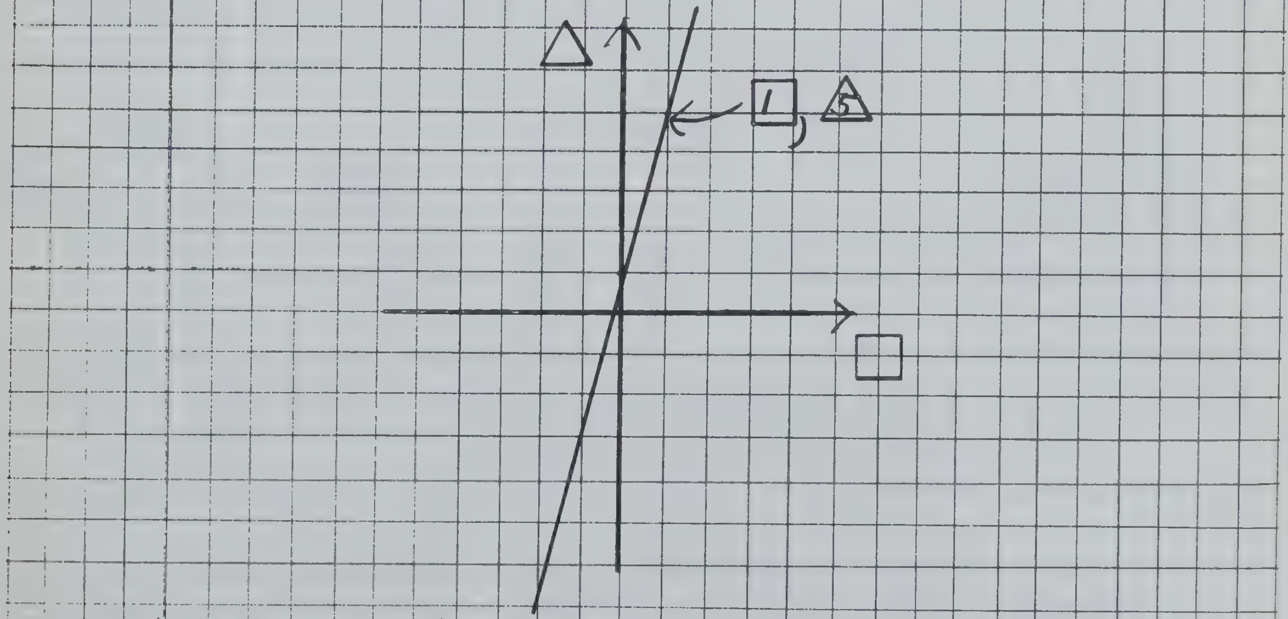
Answers:

1) The second graph is. In the first graph, the line misses the proper points, and the axes do not lie along the rulings of the paper. In the third graph, the points are too large, and the line is too thick. You can't read intersections with any precision.

2) $\triangle 1$ corresponds to $\boxed{0}$, as we can see from the second graph of problem one:



3) $\boxed{1}$, $\triangle 5$, as we can see from the second graph of problem one:



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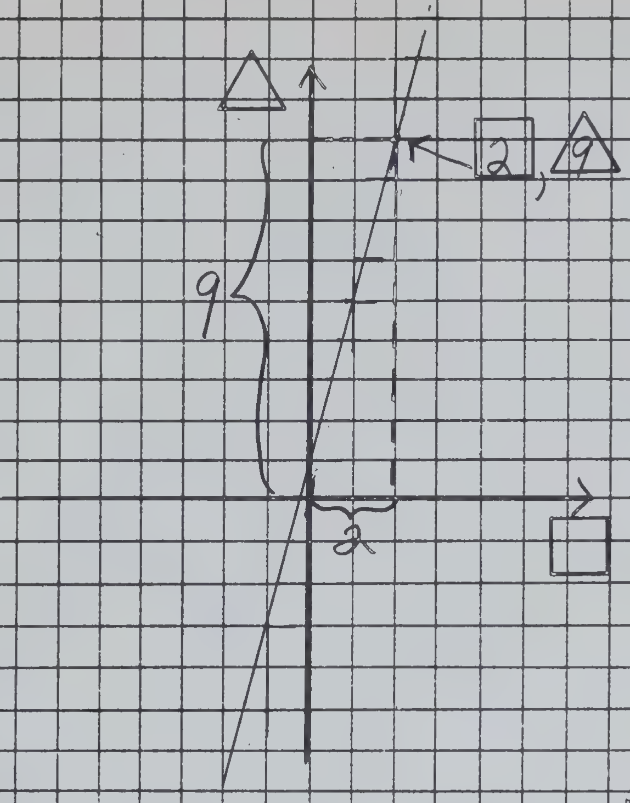
4)

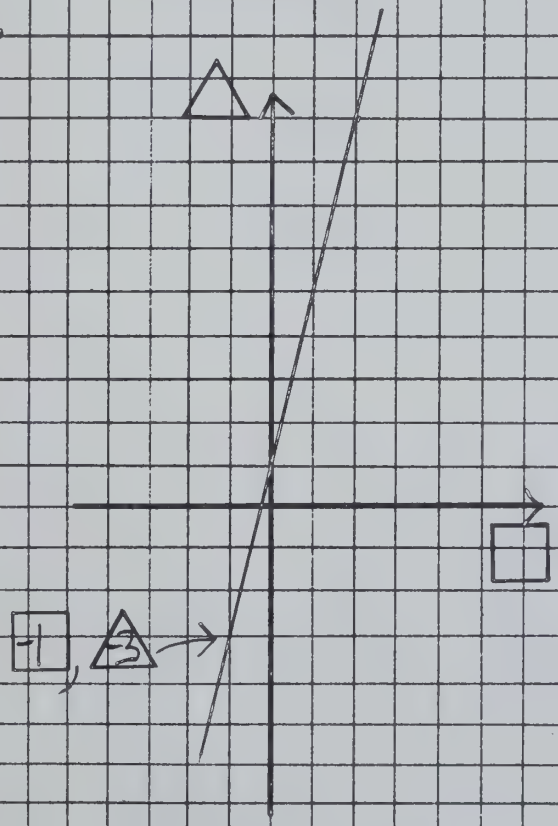
□	△
2	?



So the answer is:

□	△
2	9

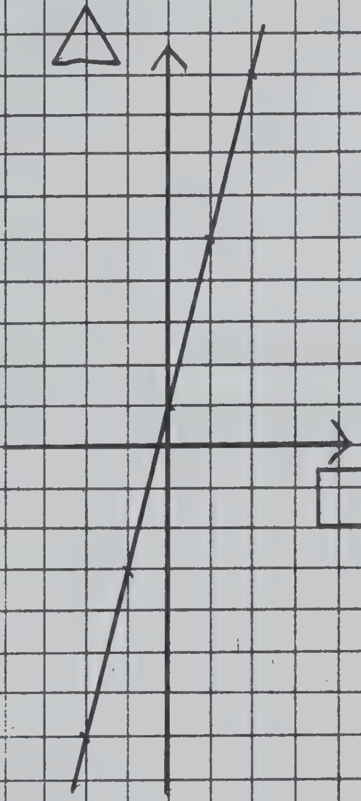
Similarly,



So we have;

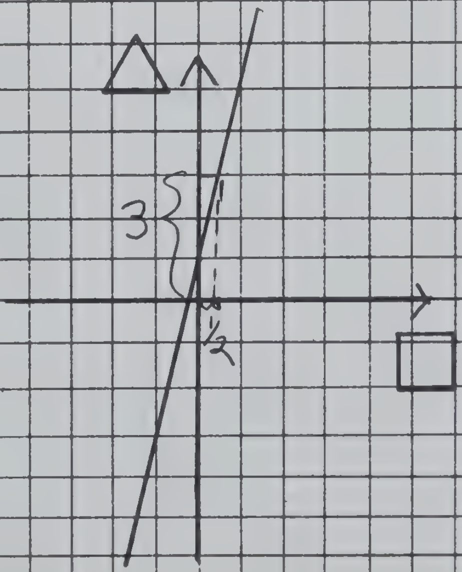
□	△
2	9
3	13
$^-1$	$^-3$

Continuing we get:



□	△
2	9
3	13
$^-1$	$^-3$
$^-2$	$^-7$

Similarly



and we have

□	△
2	9
3	13
-1	-3
-2	-7
½	3

In this same way we find that:

□	△
1½	7
-½	-1
-1½	-5

□	△
½	3
1¾	4

5) Simply substitute the numbers into □ and △ and compute.

6) - 10). These are generally similar to questions 1 - 5.

32. LETTERS FOR PLACEHOLDERS

We now begin a gradual transition from the " $\square$, $\triangle$ " notation to the " x , y" notation. For the next few months (in our own classes) we gradually make increasing use of x, y notation. (We usually give students their choice: "Do you want to do this problem with x's and y s or with boxes and triangles?" If they still mainly prefer " $\square$ " and " $\triangle$ " there is no need to hurry about changing.)

Answers:

1) $y = x + 3$
2) $\triangle = \square + 7$
3) Either $2 \cdot x$, or else $2x$.
4) a) $\triangle = (4 \times \square) + 3$
   b) $\triangle = (3 \times \square) - 2$
   c) $\square + \triangle = 15$
   d) $\triangle = \square - 5$
   e) $\triangle = (2 \times \square) + 1$
   f) $(3 \times \triangle) + (4 \times \square) - 2 = 0$
5) x^2 (read "x square" or "x squared")
6) $\triangle = \square \times \square$
7) a) $3^2 = 3 \times 3 = 9$
   b) $5^2 = 5 \times 5 = 25$
   c) $7^2 = 7 \times 7 = 49$
   d) $11^2 = 11 \times 11 = 121$
8) $\square \times \square = \triangle \times \triangle$
9) A circle of radius five, with the center at the origin.
10) cf. #9 above.
11) and 12) Similar to the preceding chapter.

33. DESCRIBING GRAPHS OF TRUTH SETS

The idea of this section is:

1) We need some good, efficient way to describe straight-line graphs;

2) An excellent method is the "y-intercept, slope" method (cf. any good book on analytic geometry);

3) Since this is the best answer to a real problem we do not need to tell the children what this method is. If, instead, we work with straight line graphs and experience a frequent need to communicate, to describe some line that we have in mind, then the children will work out for themselves a suitable method of description. By steering the class discussion slightly we can be reasonably sure that the children will discover the extremely useful "slope-intercept" form for describing straight lines.

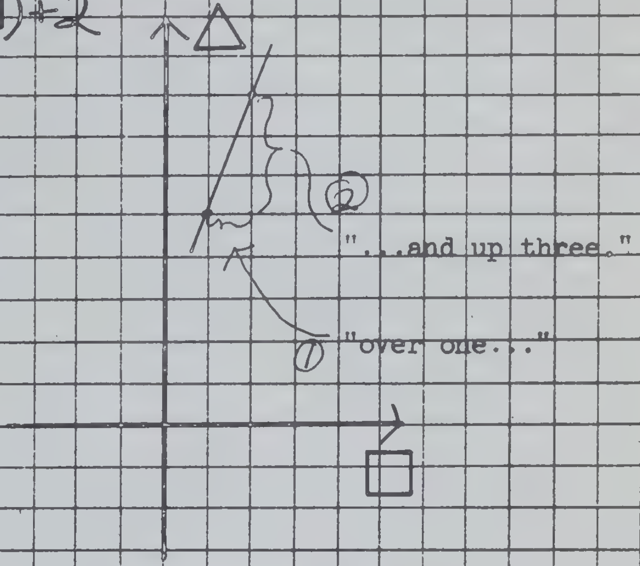
The sequence of questions in the student workbook is intended to help provoke a suitable class discussion.

34. MYRA'S DISCOVERIES

In this section we want the children to discover the "slope" and "y-intercept" coefficients, namely:

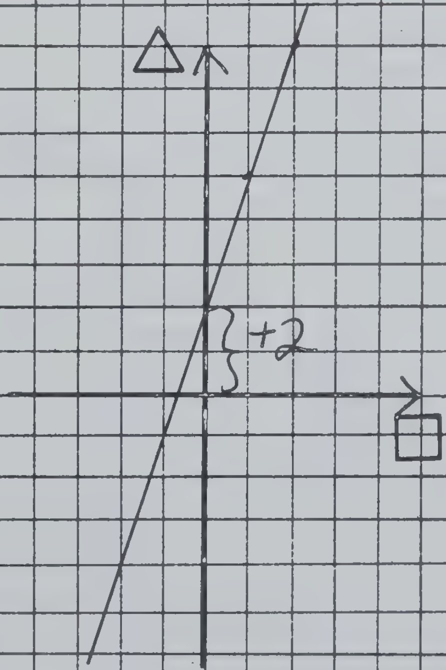
$$\triangle = (3 \times \square) + 2$$

The number that appears here in the formula determines the pattern on the graph:



The number that appears here in the formula determines the "y-intercept" on the graph:

$$\triangle = (3 \times \square) + 2$$

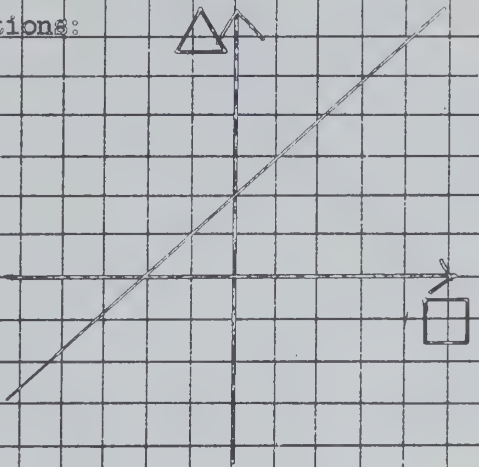


35. WRITING EQUATIONS FOR GRAPHS

This section carries further the two ideas of Section 34 --- the role of the "slope coefficient", and the role of the "y-intercept coefficient".

Answers to some of the questions:

1)



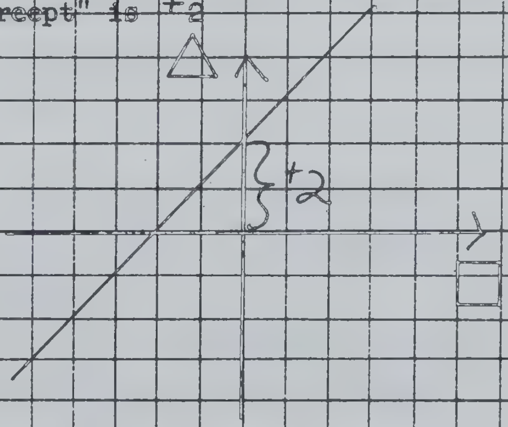
(This "answer" is not something we would ever *tell* the students! We state it here merely for the benefit of the teacher.)

First, we know that the general form of the equation must be:

$$\triangle = (\underline{\quad} \times \square) + \underline{\quad}$$

some number goes here and some number goes here.

Second, the "y-intercept" is $^{+}2$



so the number that goes here

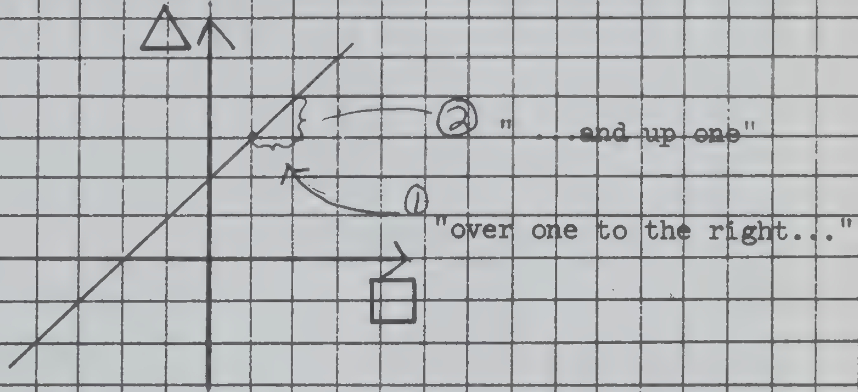
$$\triangle = (\underline{\quad} \times \square) + \underline{\quad}$$

must be ${}^+2$, i. e.,

$$\triangle = (\quad \times \square) + {}^+2$$

We still don't know what number goes here.

Finally, the "pattern" is "over one to the right, and up one"



Consequently, the number that goes here

$$\triangle = (\quad \times \square) + {}^+2$$

must be ${}^+1$, and we have

$$\triangle = ({}^+1 \times \square) + {}^+2$$

2) Paul, of course, is right.

3) Yes, John (also) is right.

4) Mary is right (cf. answer to question 1 above.)

5) No. Jerry has confused the roles of the two coefficients.

6) This, of course, is a method for deciding the answer to question #5. Carl's table is correct.

7) This continues the method of questions 5 and 6. Marie's table is right. Note that Carl's table came from Jerry's equation, whereas Marie's table came from Mary's equation.

What remains to do? We need a table from the graph, which we can compose with the two we have just made, in order to see if either Jerry or Mary is right.

8) Cf. answer to question 7 above. Mary's equation is correct.

9) Yes. Wayne's equation is correct.

10) $\triangle = ({}^{+}1 \times \square) + {}^{+}1$

11) $\triangle = ({}^{+}2 \times \square) + {}^{+}3$

12) $\triangle = ({}^{+}2 \times \square) + {}^{-}3$

13) $\triangle = ({}^{+}1 \times \square) + {}^{-}4$

14) $\triangle = ({}^{-}3 \times \square) + {}^{+}1$

15) $y = ({}^{+}3 \cdot x) + {}^{+}1$

16) $y = ({}^{+}5 \cdot x) + {}^{-}2$

17) $\triangle = ({}^{-}1 \times \square) + 0$, or simply $\triangle = {}^{-}1 \times \square$

18) $\triangle = ({}^{+}\frac{1}{2} \times \square) + {}^{-}1$

19) $\triangle = ({}^{-}\frac{1}{2} \times \square) + {}^{+}1$

20) Yes, he is.

21) Yes, Kathy is also right.

22) Yes. (Incidentally, problems with fractional slope give children a new geometric visualization of the meaning of fractions. This can be valuable in their work with "traditional" arithmetic.)

23) $y = ({}^{-}4 \cdot x) + {}^{+}2$

24) $y = ({}^{-}\frac{1}{2} \cdot x) + {}^{+}8$

25) $y = ({}^{+}\frac{1}{4} \cdot x) + {}^{+}1$ (You may want to use a sequence of questions similar to #20, #21, #22, to help the children answer #25.

26) Yes.

27) Yes.

28) "Over one to the right, and up $\frac{1}{4}$."

29) $y = ({}^{-}7 \cdot x) + {}^{+}3$

30) $\triangle = ({}^{+}5 \times \square) + {}^{-}3$ (You may want to use a sequence of questions similar to #26, #27, #28, to help the children answer question #30. We hope that by now the children will make up such a sequence of questions on their own.)

31) $\triangle = ({}^{+}\frac{3}{2} \times \square) + {}^{+}1$

33) $\triangle = ({}^{+}6 \times \square) + {}^{-}3$ (We insert this "easier" problem here in order to reinforce the student's conception of "slope-intercept" form in its basic outline.)

34) $\triangle = ({}^-\frac{5}{2} \times \square) + {}^+4$,

or else $\triangle = ({}^-2.5 \times \square) + {}^+4$

or even $\triangle = ({}^-2\tfrac{1}{2} \times \square) + {}^+4$

35) $\triangle = ({}^+\tfrac{1}{3} \times \square) + {}^+2$ (The purpose of thie "easier" problem at this point in the sequence is similar to that of question #33.)

36) $\triangle = ({}^+2 \times \square) + {}^+4$

37) $\triangle = (0 \times \square) + {}^+3$,

which can also be written

$$\triangle = {}^+3.$$

---

36. THE EQUATION OF A GRAPH

This section merely continues the work of Section 35.

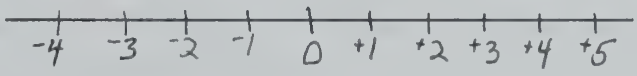
37. FINDING TRUTH SETS

This chapter is mostly review, plus a preliminary introduction to one new idea.

We do not ordinarily use this chapter as a single lesson. Rather, we use these review problems at the start of five or six lessons, and then introduce the new material in each day's lesson. (You may wish to make up many more problems of these same types.)

Questions 1-10. Answers: 1 - true; 2 - false; 3 - true; 4 - open (the truth set is $\{4\}$); 5 - false; 6 - open (the truth set is $\{{}^{+}11,\ {}^{-}11\}$ -- or, if you don't wish to stress signed numbers at this point, you might prefer to accept the answer $\{11\}$); 7 - false; 8 - true; 9 - true; 10 - true. (Remember, the symbol "$<$" means "is less than," or "is to the left of... [on the number line]." Thus $3<5$ might be read either "3 is less than 5," or else "3 lies to the left of 5 on the number line".)

Questions 11 - 22. Answers: 11 - open (The truth set is $\{4, 5, 6, 7, 8, 9, 10\}$ if we use only integers; if we use also fractions, we must include all the numbers that are greaterthan 3 and yet also less than 11, such as 3 1/2, 4 1/3, 3 2/7, 6 1/5, 10 9/17, 9 1/2, . . . In this case, there are many ways to write the truth set. One possibility is:
$\{4,5,6,7,8,9,10,\ 3\ 1/2,\ 4\ 1/3,\ 9\ 1/2,\ 5\ 1/6,\ 7\ 2/9,\ 8\ 11/13,\ 3\ 1/4,\ldots\}$
You may prefer not to bother with the case where fractions are used.); 12 - open; 13 - false; 14 - false; 15 - false; 16 - true; 17 - true; 18 - true; 19 - true; 20 - true (i. e. " ${}^{-}2$ does lie to the left of ${}^{+}1$ on the number line").



21 - false; 22 - false.

Questions 23 - 24. Answers:

23. This, of course, depends upon your class. The answer will probably look something like this:

$\{$Judy, Mary, Albert, Jonathan, Ray, Marie$\}$

24. Again, it depends... The answer will presumably look somewhat like this:

$\{$Tuesday$\}$,

or, perhaps,

$\{$March 4, 1960$\}$,

or possibly

$\{$the last day before the Christmas vacation$\}$.

Questions 25 -63.

A game: Strange as is may seem to those who have never taught the Madison Project materials, these questions are used (in our classes) mainly with third and fourth graders! With children this age, we present this material in the form of a game. We divide the class into two teams, and we write one equation at a time on the blackboard. Children raise their hands as soon as they think they have the answer, and whoever gets an answer first wins 10 points for his team.

The third and fourth graders love this game, and very frequently beg us to play it "to 200 points." (I.e., whichever team gets 200 points first, becomes the winner.)

You may think of other (and still better) ways to present this material in the classroom.

Questions 25-54. These questions are especially good with third graders; they are also good with fourth graders. (We realize that most schools do not begin the Madison Project materials this early. In our own classes we prefer to start with third or fourth graders, but you will need to work out a variation to fit your school -- beginning the study of Madison Project materials in grade seven is quite common.)

Of course, the first few times you play this game, not all students will understand how to solve every equation -- but the rivalry sooner or later stimulates most of them to get in on the act. Patience (on the part of the teacher) is required to "wait out" those children who do not at first participate. (We ourselves never urge them -- it is not necessary; we just wait.)

Answers: 25. $\{5\}$

26. $\{2\ 1/2\}$ (Our third-graders have no trouble with this -- at least, not by January or so.)

27. $\{101\}$

28. {103}
29. {2 1/2}
30. {5 1/3} (Again, our third-graders are able to do this -- thanks to Mrs. Baker's arithmetic teaching.)
31. {100 1/2} If you are surprised that our third graders find this easy -- well, so are we! But they do!
32. {5}
33. {3 1/11}
34. {2 1/7}
35. {1 3/4}
36. {2,3} Don't tell the children the secret! However, between you and me -- and don't tell! -- the idea is to find two numbers whose product is 6, and whose sum is 5.

37. {3, 4}
38. {3, 50}
39. {11, 2}
40. {5, 10}
41. {2, 100}
42. {13, 2}
43. {11, 9}
44. {6, 8}
45. {24, 2}
46. {12, 3}
47. {50, 10}
48. {1, 2}
49. {5, 0}
50. {8, 12}
51. {15, 20}
52. {75, 2}
53. {5, 3}
54. {3, 100}

Questions 55 - 63.

Grade level. We do not usually do these with third-graders. We do, however, do them with fourth-graders, and our present class (1959-60) of fourth-graders at Weston, Connecticut can solve these problems almost the instant that we write one on the board. It is hard to see whose hand went up first. (This year, the fourth-graders have learned this topic better than the older children, but that may be a result of our own unconscious emphasis.)

Answers. 55. $\{^+8, ^-3\}$

56. $\{^+5, ^-2\}$
57. $\{^+7, ^-2\}$
58. $\{^-3, ^+1\}$
59. $\{^-5, ^-3\}$
60. $\{^-5, ^-10\}$
61. $\{^-8, ^-2\}$
62. $\{^-10, ^+5\}$
63. $\{^+7, ^+3\}$

Questions 64 - 88.

This is the new topic in this chapter.

We do *not* expect all the children in class to learn this the first day. A few may -- but the real idea is to get them started thinking about this kind of equation *in advance of Chapter 38, Balance Pictures*. Call this, if you will, "readiness building" for Chapter 38.

The "secret" of these equations will be easier for most children to discover *after* they have worked with *balance pictures*.

We do not tell them any part of the secret!!

Answers. 64. $\{6\}$

65. $\{11\}$
66. $\{20\}$
67. $\{2\}$
68. $\{15\}$

Questions 69 - 75. By the time you reach Question 68, a number of students have probably discovered the rule "subtract 8 from 19 [in $8+(2\times\square)=\square+19$]". Of course this rule does not work in general. The students discover this the moment they try it on Question 69, where their "answer" is *twice* the right answer. The following sequence of problems are intended to help the students to discover a proper refinement of their first primitive rule.

Don't be disappointed if they do not discover this the first day. We shall come back to it later. (By that time we will have *balance pictures* to work with, which makes it much easier to visualize equations of this sort.)

Answers. 69. $\{6\}$

70. $\{5\}$
71. $\{4\}$
72. $\{1\}$
73. $\{100\}$
74. $\{40\}$
75. $\{35\}$

Questions 76 - 88. We now try a different tack, in the following sequence of problems.

Answers. 76. $\{8\}$

77. $\{2\}$
78. $\{10\}$
79. $\{5\}$
80. $\{12\}$
81. $\{10\}$
82. $\{2\}$
83. $\{15\}$
84. $\{6\}$
85. $\{3\}$

86. In our fourth-grade class (at Weston) this year, the children guessed numbers for this one for several minutes -- and then one girl, Beth, said, "There isn't any answer. Whatever you put in the square, one time you add 3 to it, and one time you add 5 to it, so they'll never come out equal."

Beth, of course, was exactly right!

87. $\{20\}$

88. $\{7\}$

Questions 89 - 106. At this stage, our classes are still not reliable in their responses to questions like these. Our only suggestion is to be patient. This is traditionally eleventh-grade work. We have plenty of time. (Our fourth-graders are better at this than our sixth-graders are!)

89. $\triangle = ({}^{+}1 \times \square) + {}^{+}3$
90. $\triangle = ({}^{+}2 \times \square) + {}^{+}1$
91. $\triangle = ({}^{+}1 \times \square) + {}^{+}5$
92. $\triangle = ({}^{+}3 \times \square) + {}^{-}2$
93. $\triangle = ({}^{+}3 \times \square) + {}^{-}3$
94. $\triangle = ({}^{+}1 \times \square) + 0$ $\longrightarrow$ We might also write this: $\triangle = \square$,
95. $\triangle = ({}^{-}1 \times \square) + {}^{+}3$
96. $\triangle = ({}^{-}2 \times \square) + {}^{+}1$
97. $\triangle = ({}^{-}2 \times \square) + {}^{+}4$
98. $\triangle = ({}^{+}\tfrac{1}{2} \times \square) + {}^{+}1$

If the children don't get this, you might ask them these questions:

Teacher: How would you describe the pattern?

Class: Over two the right, and up one.

Teacher: How else could you describe the pattern? [You might point to the graph, indicating "over two, up one," and then "over one, up one-half."]

Class: Over one to the right, and up one-half.

Teacher: All right, what is the equation?

(With this much help, our classes get the right answer.)

99. $\triangle = ({}^{+}1 \times \square) + {}^{+}2$
100. $\triangle = ({}^{+}\tfrac{3}{2} \times \square) + {}^{-}1$ Or else $\triangle = ({}^{+}1\tfrac{1}{2} \times \square) + {}^{-}1$

It is, of course, easy to get the $^{-}1$ on the end. The $^{+}3/2$ may not be so easy. If the class has trouble, we would go through the same sequence of questions as in Question 98.

101. $\triangle = ({}^{+}5 \times \square) + {}^{-}2$
102. $\triangle = ({}^{+}2 \times \square) + {}^{-}4$
103. $\triangle = ({}^{-}3 \times \square) + 0$
104. $\triangle = ({}^{+}\tfrac{5}{2} \times \square) + {}^{-}3$
105. $\triangle = ({}^{-}\tfrac{3}{2} \times \square) + {}^{+}2$
106. $\triangle = (0 \times \square) + {}^{+}2$ Which could also be written: $\triangle = {}^{+}2$.

38. BALANCE PICTURES

The purpose of this chapter is to give the students a new method of visualizing equations. In our classes we usually find that visualizing an equation by means of a "balance picture" is helpful in connection with transform operations, equivalent equations, and "machines" (i.e., formulas for solving equation with letter coefficients).

The idea is that we have some rolls of dimes (or washers) such as one gets at the bank. We do not know how many dimes each roll contains, and we find out by weighing the rolls.

We assume that the weight of the paper wrapper on each role is too small to effect our balance -- just as a postage stamp on a bathroom scale would read "zero".

Answers: 1) No, we cannot.

2) It must have looked like the second picture, n'est - ce pas?

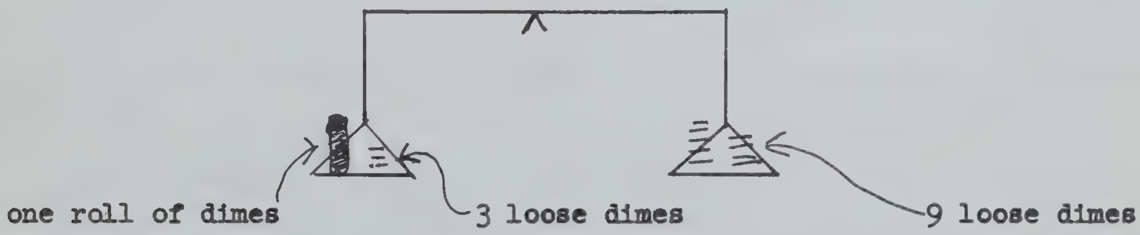
3) Now we do have enough information to tell how many dimes there are in each roll. (We assume, of course, that all three rolls contain equal numbers of dimes.)

The important thing is that the balance is in balance, and there are different combinations on the two balance pans.

In problems 4 - 7, we will gradually simplify this picture, until we finally arrive at the answer.

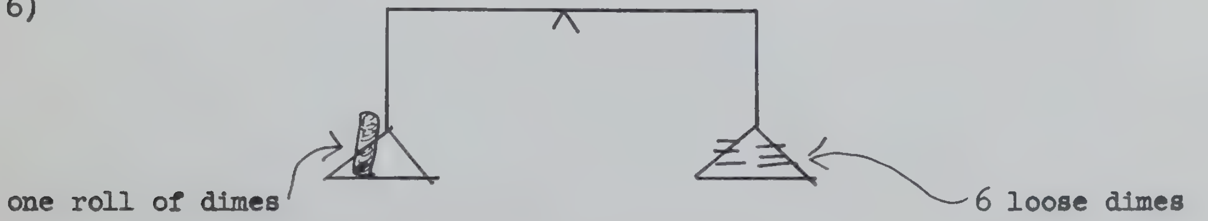
4) Yes, Sandy is right.

5)



(At this stage some children will be able to see the answer. But, let's pretend to be very stupid today, and continue step-by-step...)

6)



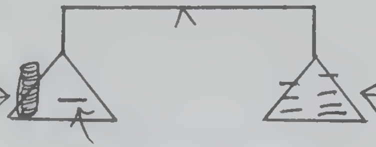
(At this stage all the children should see the answer.)

7) There must be 6 dimes in each roll.

8) He took two loose washers from each balance pan. Since Alex took the same thing from each side, the scales will still balance.

9) Yes.

10)

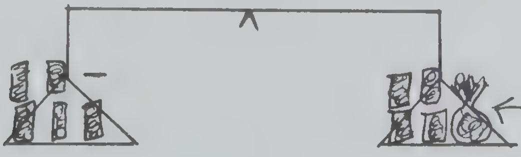


11) $(1 \times \square) + 1 = 7$, or else $\square + 1 = 7$

(Remember the identity $\square \times 1 = \square$.).

You may want to finish this problem, by taking one loose dime from each side. Evidently the truth set is $\{6\}$.

12)



13-15. Same general idea as 1-12.

16) "equivalent" equations

17) A transform operation changes the equation into an equivalent equation -- that is, it does not change the truth set.

Adding the same number to both sides of an equation is a transform operation.

Multiplying both sides of an equation by the same number is a transform operation.

Using an identity is a transform operation.

There are two others. Do you remember them?

18-21. Same idea as 1-12.

39. OPEN SENTENCES

Answers: 1) $\{7\}$

2) $\{100\ 1/2\}$ (Our third graders at Weston, Connecticut are especially fond of this kind of problem.)

3) $\{52\ 1/2\}$

4) $\{1\ 1/2\}$

5) $\{^{-}3\}$

6) $\{^{-}6\}$

7) $\{2\ 1/4\}$

8) $\{5\ 1/3\}$

9) $\{7\ 1/5\}$

10) $\{1\ 2/3\}$

11) $\{3, 2\}$

12) $\{2, 5\}$

13) $\{25, 3\}$

14) $\{7, 100\}$

15) $\{4, 2\}$

16) $\{40, 20\}$ This is usually a rather hard one. But they get it sooner or later...

17) $\{^{+}7, ^{-}2\}$

18) $\{^{+}11, ^{-}2\}$

19) $\{^{-}3, ^{-}2\}$

20) $\{^{-}7, ^{-}1\}$

21) $\{1, 2\}$

22) $\{12\}$

23) $\{3\}$

24) $\{26\}$

25) $\{5\}$

26) $\{20\}$

27) $\{2\}$

28) $\{5, 3\}$ Notice that you must use a transform operation (namely, "subtract 6 from each side of the equation") in order to get a 0 on the right hand side, before applying the coefficient rules: $3 + 5 = 8$, $3 \times 5 = 15$.

Our fourth-graders at Weston, Connecticut are especially fond of problems like this.

29) Again, we must first subtract 20 from each side, getting

$$(\square \times \square) - (17 \times \square) + 66 = 0$$

We can now apply the coefficient rules, to get: $\{11, 6\}$

30) Subtracting 75 from each side, we get:

$$(\square \times \square) - (25 \times \square) + 150 = 0$$

Hence, the truth set is: $\{15, 10\}$

31) Subtracting 202 from each side, we get:

$$(\square \times \square) - (8 \times \square) + 12 = 0$$

Consequently, the truth set must be: $\{2, 6\}$

32) This equivalent to

$$(\square \times \square) - (13 \times \square) + 40 = 0$$

so the truth set is $\{8, 5\}$

33) $\{3\}$

34) $\triangle = (^{+}1 \times \square) + {}^{+}2$ or else $\triangle = \square + 2$

35) $\triangle = (^{+}2 \times \square) + {}^{+}3$

36) $\triangle = (^{+}1 \times \square) + {}^{-}3$

37) $\triangle = (^{+}\frac{1}{2} \times \square) + {}^{+}1$

38) $\triangle = (^{-}2 \times \square) + {}^{+}4$

39) $\triangle = (^{+}1\frac{1}{2} \times \square) + {}^{+}2$

or else $\triangle = (^{+}\frac{3}{2} \times \square) + {}^{+}2$

or else $\triangle = (^{+}1.5 \times \square) + {}^{+}2$

40) $\triangle = (^{+}3 \times \square) + 0$

or else $\triangle = (^{+}3 \times \square)$

or else $\triangle = 3 \times \square$

41) $\triangle = (0 \times \square) + {}^{+}3$

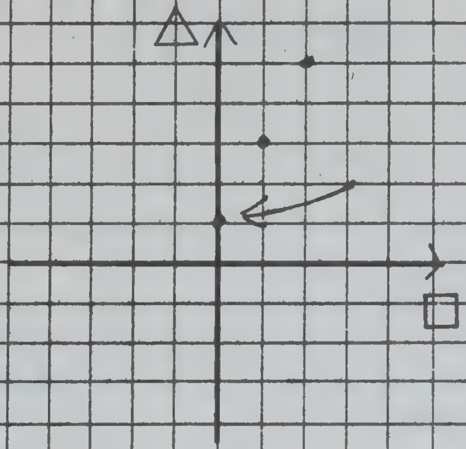
or else $\triangle = {}^{+}3$

or else $\triangle = 3$

42) $\triangle = (^{-}1 \times \square) + {}^{-}1$

43) $\triangle = (^{+}2\frac{1}{2} \times \square) + {}^{-}3$

44) First, we can extend the "pattern" to find where the line will cross the y-axis:



Now, we see that the equation must be

$$\triangle = ({}^{+}2 \times \square) + {}^{+}1.$$

45--59. Similar to those in Chapter 38.

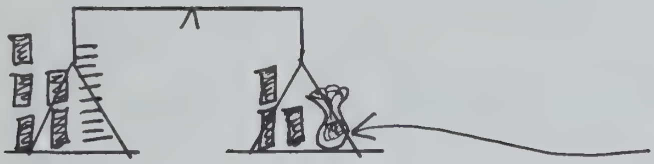
40. CINDY'S MACHINE

Answers:

1) $\{6\}$

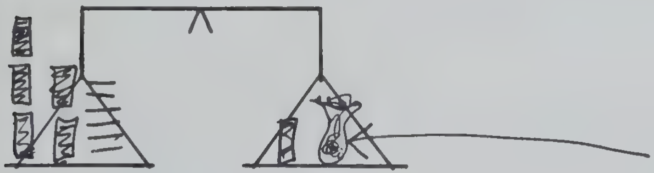
2) $\{2\}$

3)



bag with 24 loose washers or dimes

4)



bag with 18 loose washers

5) "equivalent" equations

6) one which does not change the truth set

7) $\{4\}$

8) $\{3\}$

9) $\left\{\frac{b}{a}\right\}$

10) $\square = \frac{b}{a}$

11) We must see the relation between

$$a \times \square = b$$

and

$$5 \times \square = 30$$

Evidently for "a" we must write "5", and for "b" we must write "30".

Cindy's machine, then, becomes $\square = \frac{30}{5}$,

which leads to the right answer: $\{6\}$

12-17. These should be easy.

18) $\square = \frac{c-b}{a}$ The children may have to guess quite a few times before they hit on this.

19) No.

20) No.

21) - 23) We hope this sequence will help get the children on the right track.

24) Now we try again; of course, the answer is still $\square = \frac{c-b}{a}$

25) $\square = \frac{b}{c-a}$

26) - 28) We hope this will provide a clue to problem #25.

29) $\square = \frac{c}{a+b}$

30) $\square = \frac{c+d-b}{a}$

31) $\square = b-a$

32) $\square = \frac{d-a-b}{c}$

or else $\square = \frac{d-(a+b)}{c}$

33) $\square = \frac{c+d}{a+b}$

34) $\square = \frac{(a+b)-(d+e)}{c}$

35) $\square = \frac{d-b}{a-c}$

36) $\square = \frac{a}{b}$

37) $\square = \frac{a}{b+c}$

38) $\square = \frac{b+c+d}{a}$

39) $\square = \frac{d}{a+b+c}$

40) $\square = \frac{c}{a+b-d}$

41) $\square = \frac{c}{d-(a+b)}$

or else $\square = \frac{c}{d-a-b}$

41. DERIVATIONS

This lesson -- actually, this is a sequence of four or five lessons-- is a favorite with our fifth grade class at Weston. Children find this a bit hard at first (And no wonder! It is hard.), but once they learn how to make derivations this is an accomplishment they are proud of.

Answers:

1) An identity is an open sentence that becomes true for every correctly made substitution. (No substitution will make it false.)

2) Here are a few:

$\square + 3 = 3 + \square$

$\square + 4 = 4 + \square$

$\square \times (3 + \triangle) = (3 \times \square) + (\square \times \triangle)$

$\square \times \triangle = \triangle \times \square$

$A + (B \times C) = (C \times B) + A$

$\square + \triangle = \square + \triangle$ (A "trivial" identity -- i.e. both sides are exactly the same)

$\square + 3 = \square + 3$ (Another "trivial" identity)

$\square + 0 = \square$

$\square \times 1 = \square$

$\square \times 0 = 0$

$(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$

$(\square + 5) \times (\square + 5) = (\square \times \square) + (10 \times \square) + 25$

$A \times (B + 7) = (7 \times A) + (A \times B)$

$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + [(\triangle + \triangle) \times \square] + (\triangle \times \triangle)$

$\square + \square = 2 \times \square$

$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$

$\square + \square + \square = 3 \times \square$

$(A + B) \times (C + D) = (A \times C) + (A \times D) + (B \times C) + (B \times D)$

$\square + 1961 = 1961 + \square$

$\square + 1962 = 1962 + \square$

Obviously, you can go on to make up as many more as you like, by following these patterns, or even by finding some new ones.

3) This is a provocative question. There are two kinds of answers that show understanding.

The expected answer (i.e., the one I expected) is: "as many as you care to write down" (or, "an infinite number," or "there's no end to the list"). But my Weston firth-graders surprised me by answering "about 5 or 6." This also shows real understanding. When I asked them if there weren't more than just "5 or 6," they said yes, but these 5 or 6 were all you really needed to know. The others were just "saying the same thing, with two or three of these basic ones combined together."

I was delighted with this answer. I have never heard a better statement of the role of axioms in mathematics!

4) These are (at least roughly) the "basic identities" that are "all that we need to know." (Technically, they are -- roughly -- the "field axioms.")

5) Here are the names:

$\square + \triangle = \triangle + \square$ commutative law for addition (CLA)

$\square \times \triangle = \triangle \times \square$ commutative law for multiplication (CLM)

$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$ distributive law (DL)

$\square \times 1 = \square$ law of one (L1)

$\square + 0 = \square$ law of zero (Lz)

$\square \times 0 = 0$ (we use no name for this one--you could make one up if you wanted)

$\frac{\square}{\square} = 1$ (again, we use no name for this one)

$\square = \square$ This is one example of a "trivial" identity.

$$\left.\begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ \quad\vdots \end{array}\right\}$$

We call these "changing names" (CN). That is, "2" is just another name for "1 + 1".

6) In a derivation, we start with a trivial identity and change it one step at a time until we end up with the identity that we want.

7) Three possible answers: "They prove that an identity is really correct."

"They prove that a statement is true."

"They show us that we can cross the identity off our list, since it is just a combination of other identities."

(The third answer above is the most satisfactory. Still other answers are possible, and would show understanding.)

8) In this, and the following derivations, we always begin with a "trivial" identity, and end with the identity we wanted to "prove". The format used in the following derivations is the one used by our fifth-grade students at Weston, Connecticut. Several of these lessons are available in tape recordings and studying these tapes may be helpful. A valuable reference is: THE LOGIC OF ELEMENTARY MATHEMATICS, by R. M. Exner and M. Rosskopf (McGraw-Hill 1959).

Theorem: $A+(B\times C)=(C\times B)+A$

Proof: $A+(B\times C)=A+(B\times C)$ "trivial"

$A+(B\times C)=(B\times C)+A$ commutative law for addition, by substituting $A\to\Box$ and $B\times C\to\triangle$

$A+(B\times C)=(C\times B)+A$ CLM, $B\to\Box$, $C\to\triangle$

q.e.d.

9) It is probably clear that these "derivations" are a challenge of the teacher's best efforts -- but the results can be delightfully rewarding.

Theorem: $(A+B)\times(A+B)=(A\times A)+[(B+B)\times A]+(B\times B)$

Proof: $(A+B)\times(A+B)=(A+B)\times(A+B)$ We begin with the "trivial" identity.

$(A+B)\times(A+B)=[(A+B)\times A]+[(A+B)\times B]$ This was obtained from the distributive law,

$$\Box\times(\triangle+\nabla)=(\Box\times\triangle)+(\Box\times\nabla)$$

by substituting $A+B\to\Box$, $A\to\triangle$, $B\to\nabla$.

$(A+B)\times(A+B)=[A\times(A+B)]+[(A+B)\times B]$ The reason for this is: this was a "tactical" step, to get the common factor "A" on the left so that we can use the distributive law again.

The logical justification is:

commutative law for multiplication;

$\square \times \triangle = \triangle \times \square$,

$A \rightarrow \square$, $A+B \rightarrow \triangle$

$$(A+B)\times(A+B) = (A\times A)+(A\times B)+[(A+B)\times B]$$

Justification: distributive law,

$A \rightarrow \square$, $A \rightarrow \triangle$, $B \rightarrow \bigtriangledown$

(We now use on the second brackets the same technique we just used on the first bracketed term.)

$$(A+B)\times(A+B) = (A\times A)+(A\times B)+[B\times(A+B)]$$

Justification: CLM

$$(A+B)\times(A+B) = (A\times A)+(A\times B)+(B\times A)+(B\times B)$$

Justification: DL

$$(A+B)\times(A+B) = (A\times A)+(A\times B)+(A\times B)+(B\times B)$$

Justification: CLM

Reason: We want to get $[(B+B)\times A]$, so A must be the common factor. In the DL, the common factor must appear on the left. (This is the hardest step in any of these derivations. You will probably need to help the class with this step.

$$(A+B)\times(A+B) = (A\times A)+[A\times(B+B)]+(B\times B)$$

Justification: Distributive law (used "backwards," as our fifth graders say),

$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$,

$A \rightarrow \square$, $B \rightarrow \triangle$, $B \rightarrow \bigtriangledown$

$$(A+B)\times(A+B) = (A\times A)+[(B+B)\times A]+(B\times B)$$

Q.E.D.

Justification: CLM, with

$B+B \rightarrow \square$, $A \rightarrow \triangle$

10) We write the "justification" portion somewhat more briefly:

Theorem: $2 \times 3 = 3 + 3$

Proof: $2 \times 3 = 2 \times 3$ "trivial"

$2 \times 3 = (1 + 1) \times 3$ "changing names", i.e., $1 + 1 = 2$

$2 \times 3 = 3 \times (1 + 1)$ CLM

$2 \times 3 = (3 \times 1) + (3 \times 1)$ DL

$2 \times 3 = 3 + (3 \times 1)$ L 1

$2 \times 3 = 3 + 3$ L 1

q. e. d.

11) Theorem: $2 \times 3 = 6$

Proof: $2 \times 3 = 2 \times 3$ "trivial"

$2 \times 3 = (1 + 1) \times 3$ "changing names" (CN)

$2 \times 3 = 3 \times (1 + 1)$ CLM

$2 \times 3 = (3 \times 1) + (3 \times 1)$ DL

$2 \times 3 = 3 + (3 \times 1)$ L 1

$2 \times 3 = 3 + 3$ L 1

$2 \times 3 = 3 + (2 + 1)$ CN

$2 \times 3 = 3 + 1 + 2$ CLA

$2 \times 3 = 4 + 2$ CN (namely, 3   1   4)

Note: We are avoiding explicit use of the associative laws at this point. We will introduce them a few lessons later on.

$2 \times 3 = 4 + 1 + 1$ CN $(1 + 1 = 2)$

$2 \times 3 = 5 + 1$ CN $(4 + 1 = 5)$

$2 \times 3 = 6$ CN ("changing names")

Q. E. D.

12) and 13) Similarly.

14) $(A+B+C) \times (D+E+F) = (A \times D)+(A \times E)+(A \times F)+(B \times D)+(B \times E)+$
$(B \times F)+(C \times E)+(C \times D)+(C \times F)$.

42. THE "ASSOCIATIVE" LAWS

This section follows the same idea as the preceding section, but we now introduce in explicit form the two associative laws:

$$\square + (\triangle + \nabla) = (\square + \triangle) + \nabla \qquad (ALA)$$

$$\square \times (\triangle \times \nabla) = (\square \times \triangle) \times \nabla \qquad (ALM)$$

Several class lessons with fifth graders doing these problems have been tape-recorded, and will be available in the near future.

- -

43. AREA

44. HOW MANY SQUARES?

These two lessons have worked very well with our fourth-grade students at Weston, Connecticut.

45. SIMULTANEOUS EQUATIONS

This is the most "experimental" chapter in this book. It is the only chapter that is not yet fully tested.

I, personally, am not satisfied with my own teaching of this topic. I include it in the present volume for two reasons:

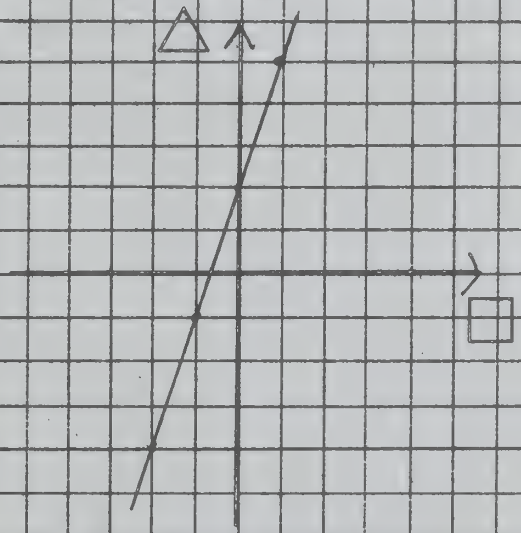
1) Several teachers in Syracuse, New York, who have worked with this topic for two or three years, have developed some very effective ways to present it, so the topic itself is apparently not hopeless;

2) The school mathematics program nowadays is -- and, I believe, should be -- quite fluid. I want each teacher who uses this book to be a partner in our experimentation. It is neither wise nor fair for one person to develop a program, and then hand it to someone else to teach.

I have not yet found the best way to present this topic. I wish you luck in your attempts!

(A tape-recording of one of my lessons -- not entirely successful! ---- will soon be available, in case you can get any ideas from it.)

Answers: 1)



2) and 3) These are familiar to you by now, I imagine. We begin this lesson with these three questions on graphs so that the topic will be fresh in the students' minds. Graphs provide the easiest method for fifth and sixth graders to use in solving simple pairs of simultaneous equations.

4) He knows that when he substitutes $(^+2, ^+8)$ into the $\square, \triangle$ in

$$\triangle = (^+3 \times \square) + ^+2,$$

the resulting statement will be true.

How does Lex know this? Note that he need not perform the arithmetic!

Since the graph represents the truth set for the equation, every point on the graph (e.g., $(^+2, ^+8)$) will yield a pair of numbers that, when substituted into $\square, \triangle$ will produce a true statement.

(As in the "Point-Set Game", the first number of the pair must go into the $\square$, and the second number of the pair must go into the $\triangle$.)

5) Yes, for a reason similar to #4.

6) Graph each line. Do you know the secret?

7) - 17) Similarly.

46. A LAW OF PHYSICS

Most of the lessons in this book I learned from my students -- by experimenting. This lesson is an exception; I learned it from Professor Robert Karplus, a physicist from Harvard University who is now a member of the Physics Department at the University of California in Berkeley, California.

Those of you who like to play with "gadgets" may find this lesson very enjoyable.

* *

CONCLUSION

Lesson 46 is the last lesson in the present book.

Many teachers are currently experimenting with the Madison Project material, either for:

i) gifted students in grades 3 - 6;
or for ii) all kinds of students in grades 7 and 8;
or for iii) slower students in grades 9 and 10.

I am always pleased to hear from teachers who are working with this material, or with other similar material. To the best of our ability, we like to help other teachers in any way we can.

A series of tape - recordings of actual classroom lessons, accompanied by film strips, will be available in the near future. This may answer many questions.

In addition, a revised (and much improved) version of the "Workbook" and "Teachers' Manual" will be published next season by Addison-Wesley, Inc., Reading, Massachusetts.

Correspondence concerning the Madison Project is always welcome, and can be addressed to:

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Mathematics Department
Syracuse University
Syracuse 10, New York.

Financial support for some of the Project's experiments has been provided by the Marcel Holzer Foundation of New York City and the Carrier Corporation of Syracuse, New York. I should like to express my personal appreciation for this financial assistance, without which the Project would have been impossible, and also for the magnificent co-operation of a large group of hard-working teachers and supervisors in several dozen communities, including:

Syracuse, New York
Weston, Connecticut
Tarrytown, New York
North Haven, Connecticut
West Canada Valley, New York
Binghamton, New York
West Hartford, Connecticut

and a private school in New Haven, Connecticut.

Further workbook material is currently being developed, tried, and modified. It will gradually become available in a form similar to the present volume.

I wish you every success in your work with this material. I hope you will find it as much fun as we have!

1. Equations, Open Sentences, and Inequaliti

Can you fill in the proper number in the squares?

1) $3 + \square = 5$

2) $2 + \square = 6$

3) $5 + \square = 11$

4) $3 + \square = 16$

5) $8 + \square = 21$

6) $3 + (2 \times \square) = 7$

7) $4 + (2 \times \square) = 10$

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8) $3 + (3 \times \square) = 18$

9) $2 + (3 \times \square) = 14$

10) $1 + (2 \times \square) = 6$

11) $1 + (2 \times \square) = 4$

12) $2 + (2 \times \square) = 11$

Sentences or statements may be *true*, *false*, or *open*.

"Washington was once President of the United States." This is a *true* statement.

"Ann Todd was once President of the United States." This is a false statement.

"____________ is teacher of this class." This statement is neither true nor false. It contains a placeholder, namely "_______." We can fill in the "_______" so as to make the sentence true, or we can fill in the "_______" so as to make

the sentence false.

If we have not yet filled in the "__________", the statement would be called open.

Are the following statements true, false, or open?

13) $2 + 7 = 9$

14) $3 \times 4 = 18$

15) $23 \times 11 = 253$

16) $2 + \square = 5$

17) ______ sits in the front row.

18) $1 + (3 \times \square) = 7$

19) Isaac Newton discovered the theory of gravitation.

20) Karl Marx was a famous Russian writer.

21) Before the Braves moved to Milwaukee, they used to play in Boston.

22) $3 + (2 \times \square) = 27$

23) There are ____ students in this class.

24) $25 \times 15 = 175$

When we talk about the "truth set" of an open sentence, we mean the set of numbers or names that we can substitute for the placeholder to get a true statement.

Can you find the truth set for each open sentence?

25) $3 + \square = 11$

26) $1 + (2 \times \square) = 7$

27) ______ sits nearest to the door.

28) The tallest boy in class is ______.

29) ______ is teacher of

this class.

30) ________ sits in the front row.

31) $3 + \square = 31$

32) $2 + (8 \times \square) = 26$

33) $1 + (2 \times \square) = 7$

The symbol "<" means "less Than".

$3 < 5$ is a true statement.

$7 < 2$ is a false statement.

Can you find the truth sets for each sentence? (Use only whole numbers.)

34) $1 < \square < 5$

Ans. $\{2, 3, 4\}$

35) $1 < 2 + (3 \times \square) < 10$

36) $3 + (2 \times \square) = 7$

37) $3 < 1 + (2 \times \square) < 12$

38) $3 \times \square = 27$

39) $5 < 1 + (2 \times \square) < 16$

2. Graphs and Tables of Truth Sets

Can you find the truth sets for these open sentences?

1) $1 + (3 \times \square) = 16$

2) $3 + \square = 7$

3) $2 + \square = 3$

4) $1 + (2 \times \square) = 15$

5) $1 + (2 \times \square) = 8$

6) $1 + (3 \times \square) = 5$

Sometimes an open sentence

contains two placeholders. In that case, the truth set consists of ordered pairs of elements:

For example:

$$\triangle = \square + 3$$

Suppose we agree to use the order $(\square, \triangle)$ — that is, we write the numbers in the square first, and then the number to be substituted into the triangle.

7) Does (2, 5) belong to the truth set of the open sentence

$$\triangle = \square + 3 \text{ ?}$$

8) For the sentence $\triangle = \square + 3$, fill in the blank spots in this table of the truth set:

$\square$	$\triangle$
1	4
2	
3	6
4	
5	
6	9
	10
	11
$\frac{1}{2}$	
$1\frac{1}{2}$	~~$6\frac{1}{2}$~~ $4\frac{1}{2}$
$3\frac{1}{2}$	$6\frac{1}{2}$

9) Fill in the blank spots in this table of the truth set of the sentence $\triangle = \square + 5$;

$\square$	$\triangle$
1	
2	
3	
4	
	15
	20
1½	
	23½

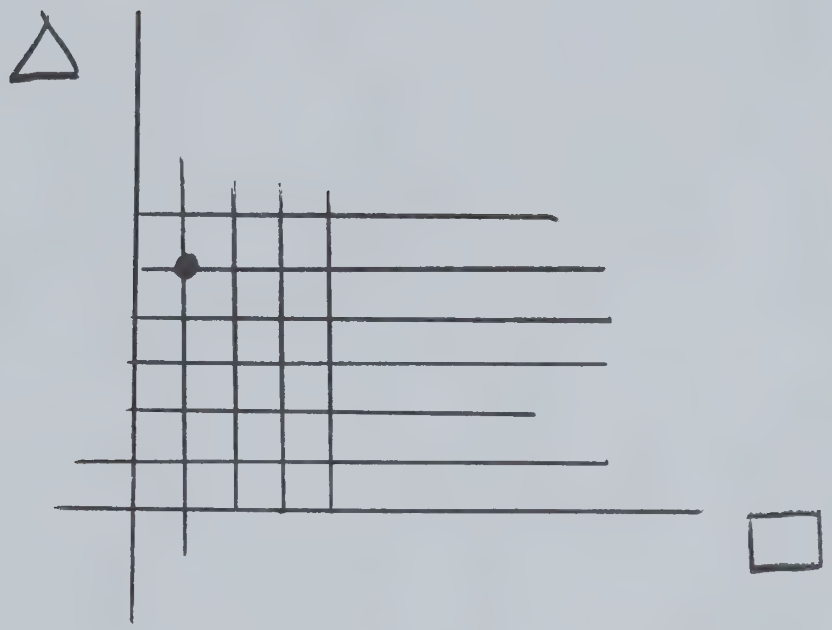
10) Can you fill in the blank spots in this table of the truth set of the sentence $\triangle = (2 \times \square) + 1$?

$\square$	$\triangle$
1	
2	
3	
4	
	13
	15
	17
$\frac{1}{2}$	
$2\frac{1}{2}$	
	8
	10

Another way of writing truth sets is to use a graph. We locate the number in the □ by counting to the right, and the number in the △ by counting up.

Example: For the sentence $\triangle = \square + 4$, (1, 5) belongs to the truth set. We would show this on a graph by counting 1 unit to the right, and five units up, and

marking the spot with some kind of mark:



Element of the Truth Set of $\triangle = \square + 4$

11) Mark several more points indicating elements of the truth set of $\triangle = \square + 4$.

12) Graph several elements of the truth set of $\triangle = \square + 2$.

13) Graph $\triangle = \square + 1$.

14) Graph $\triangle = (3 \times \square) + 1$

15) Graph $\triangle = (4 \times \square) + 1$

16) Graph $\triangle = (2 \times \square) + 1$

17) Graph $\triangle = (5 \times \square) + 1$

18) Graph $\triangle + \square = 15$

3. The Point-Set Game

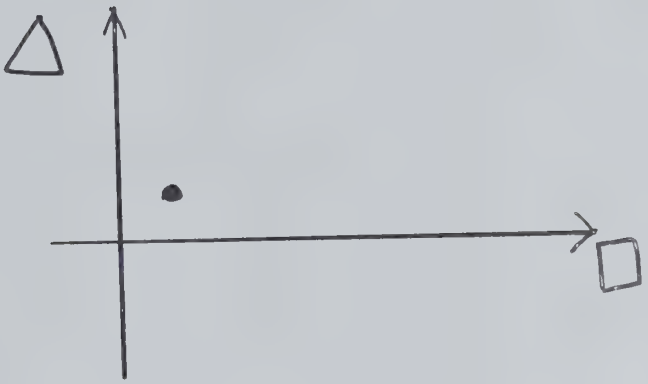
The rules: To play, you may choose any point where two lines on your graph paper intersect.

If you completely surround some of your opponent's points so that he has no "breathing space", you erase all of his points inside.

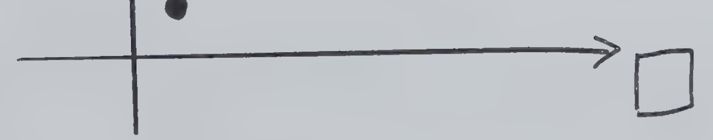
At the end of the game the person with the most points wins

Example: 1st Move: Player A chooses

$\boxed{1}$, $\bigtriangleup\!\!\!\!\!{\scriptstyle 1}\;$ ● marks A's points



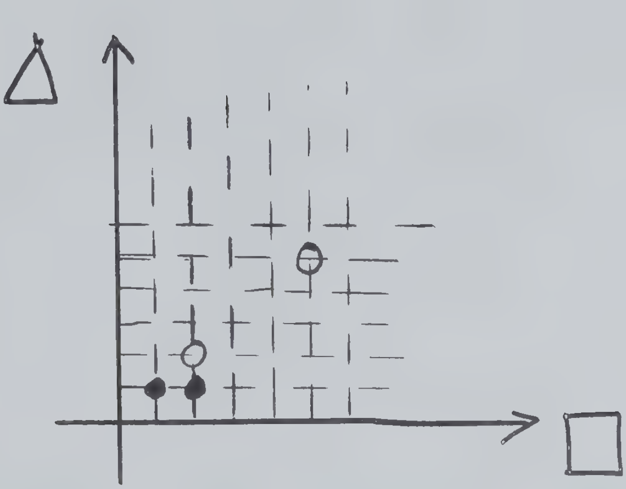
Player B chooses $\boxed{2}$, $\bigtriangleup\!\!\!\!\!{\scriptstyle 2}\;$.



○ marks B's points.

2nd move: A chooses $\boxed{2}$, $\bigtriangleup\!\!\!\!\!{\scriptstyle 1}\;$, and B

chooses $\boxed{5}$, $\triangle 5$,



3rd Move: A chooses $\boxed{1}$, $\triangle 2$; B chooses $\boxed{6}$, $\triangle 6$.

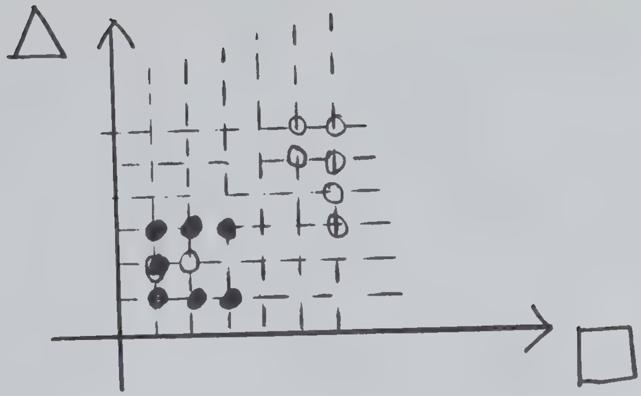
4th Move: A chooses $\boxed{1}$, $\triangle 3$; B chooses $\boxed{5}$, $\triangle 6$

5th Move: A chooses $\boxed{2}$, $\triangle 3$; B chooses $\boxed{6}$, $\triangle 5$

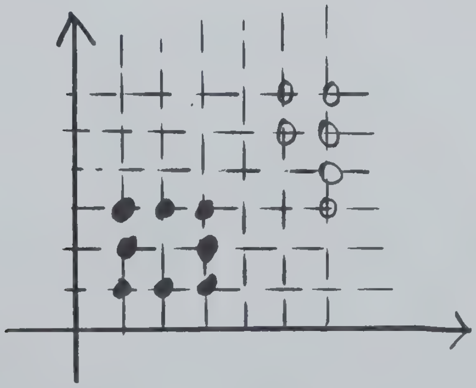
6th Move: A chooses $\boxed{3}$, $\triangle 1$; B chooses $\boxed{6}$, $\triangle 4$.

7th Move: A chooses $\boxed{3}$, $\triangle 3$; B chooses $\boxed{6}$, $\triangle 2$.

The picture now looks like this:

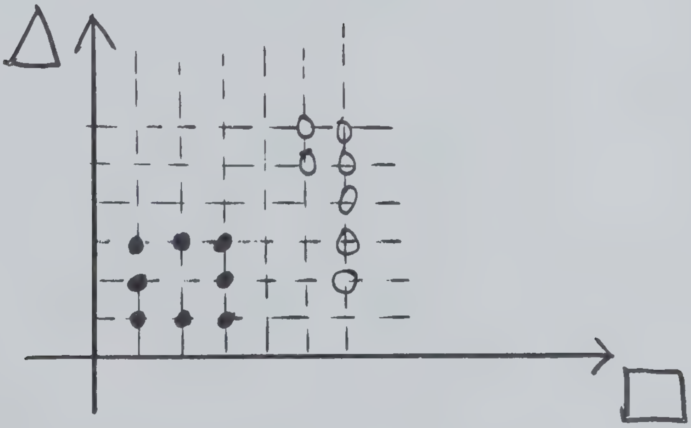


8th Move: A chooses $\boxed{3}$, △2. A has completely surrounded B's point at $\boxed{2}$, △2, so this point must be erased. The picture now looks like this:



B has not yet played. Suppose that, for his 8th move, B chooses $\boxed{6}$, $\triangle 2$.

The picture would look like this:



Questions:

1. At the end of 8th move, what is the score?

2. What mistake did B make?

4. Review

Can you tell which sentences are true, which are false, and which are open?

1) $2 + 7 = 9$

2) $3 \times 11 = 35$

3) $2 \times 9 = 18$

4) $3 \times 9 = 27$

5) $3 \times \square = 36$

6) John sits in the front row.

7) ______ sits near the windows.

8) Shirley sits near the door.

9) "______" is Governor of the State of (______).

10) $21 + \square = 32$

11) $5 \times \square = 40$

12) $19 + 13 = 22$

Can you find the truth set for each open sentence?

13) ______ sits in the front row.

14) $5 + \square = 12$

15) $1 + (3 \times \square) = 22$

16) $1 + (2 \times \square) = 6$

17) $1 + (3 \times \square) = 5$

18) $1 + (3 \times \square) = 8$

19) $1 + (3 \times \square) = 12$

20) $5 + (2 \times \square) = 11$

21) $5 + (3 \times \square) = 8$

Can you make a table for the truth set for each open sentence?

22) "______" sits in the (____) row.

23) $\triangle = 5 + \square$,

24) $\triangle + (2 \times \square) = 18$

Can you find the truth set for each open sentence?

25) $1 < 3 + (2 \times \square) < 10$

use only whole numbers

26) $1 < 3 + (2 \times \square) < 10$

use whole numbers and fractions

27) $3 < 1 + (2 \times \square) < 12$

use only whole numbers

28) $3 < 1 + (2 \times \square) < 12$

use whole numbers and

also fractions

29) $5 < 1 + (2 \times \square) < 11$

use only whole numbers

30) $5 < 1 + (2 \times \square) < 11$

use whole numbers and

also fractions

Can you graph the following equations?

31) $\triangle = \square + 3$

32) $\triangle = (2 \times \square) + 3$

33) $\triangle = (4 \times \square) + 3$

34) $\triangle = (5 \times \square) + 3$

35) $\triangle = (3 \times \square) + 3$

36) $\square + \triangle = 20$

5. More Graphs

Can you graph these equations?

1) $\triangle = (2 \times \square) - 1$

2) $\triangle = (3 \times \square) - 1$

3) $\triangle = \square - 1$

4) $\triangle = (5 \times \square) - 1$

5) $\triangle = \square$

6) $\triangle = \square + 1$

7) $\triangle = \square + 2$

8) $\triangle = \square + 3$

9) $\square + \triangle = 14$

6. Identities

Some open sentences permit us to substitute any numbers we wish. If we do our substituting correctly, the resulting statement is always *true*. Such equations are often called *identities*.

Which of the following are identities?

1) $\square + \triangle = \triangle + \square$

2) $\square \times \triangle = \triangle \times \square$

3) $\square + \triangle = 15$

4) $(\square + \triangle) \times (\square - \triangle) =$

$(\square \times \square) - (\triangle \times \triangle)$

5) $(\square \times \triangle) + (\square - \triangle) =$

$(\square + \triangle) - (\square \times \triangle)$

6) $(\square + 11) + \triangle = (11 + \triangle) + \square$

7) $(\square + 11) \times \triangle = (11 + \triangle) \times \square$

8) $3 \times (\square + \triangle) =$

$(3 \times \square) + (3 \times \triangle)$

9) $5 \times (\square + \triangle) =$

$$(5 \times \square) + (5 \times \triangle)$$

10) $6 \times (\square + \triangle) =$

$$(6 \times \square) + (6 \times \triangle)$$

11) $\bigtriangledown \times (\square + \triangle) =$

$$(\bigtriangledown \times \square) + (\bigtriangledown \times \triangle)$$

12) $\square \times (3 + \triangle) =$

$$(\square \times 3) + (\square \times \triangle)$$

13) $\square \times (\triangle + 1) =$

$$(\square \times \triangle) + (\square \times 1)$$

14) $(\square + 7) \times \triangle = (7 + \triangle) + \square$

15) $3 + \square = \square + 3$

16) $(7 \times \triangle) + \square = \square + (7 \times \triangle)$

17) $(2 \times \triangle) + \square = \square + (2 \times \triangle)$

18) $\square \times (\triangle + \triangledown) =$

$(\square \times \triangle) + (\square \times \triangledown)$

19) $(7 + \triangle) \times \square = (\square \times 7) + \triangle$

20) $(5 \times \square) + (2 \times \triangle) =$

$(2 \times \triangle) + (5 \times \square)$

21) $\square \times (\triangle + 4) =$

$$(\square \times \triangle) + (\square \times 4)$$

22) John says that the sentence

$$(3 + \triangle) \times \square = (\square + 3) \times \triangle$$

is an identity because if you substitute 1 into the square and 1 into the triangle, you get

$$(3 + \overset{1}{\triangle}) \times \boxed{1} = (\boxed{1} + 3) \times \overset{1}{\triangle}$$

$$4 \quad \times \quad 1 \quad = \quad 4 \quad \times \quad 1$$

$$4 = 4,$$

which is true.

Do you agree with John?

23) Marie says that John has not proved that the sentence is an identity. To prove it is an identity, you must prove that every substitution will produce a true statement, not merely that (1, 1) will.

Do you agree with Marie?

24) Do *you* think that the sentence

$$(3 + \triangle) \times \square = (\square + 3) \times \triangle$$

is an identity? Can you *prove* that you are right?

7. Numbers With Signs

1) Mathematicians sometimes use numbers with signs, such as:

$^{+}3$ ("positive three")

$^{-}2$ ("negative two")

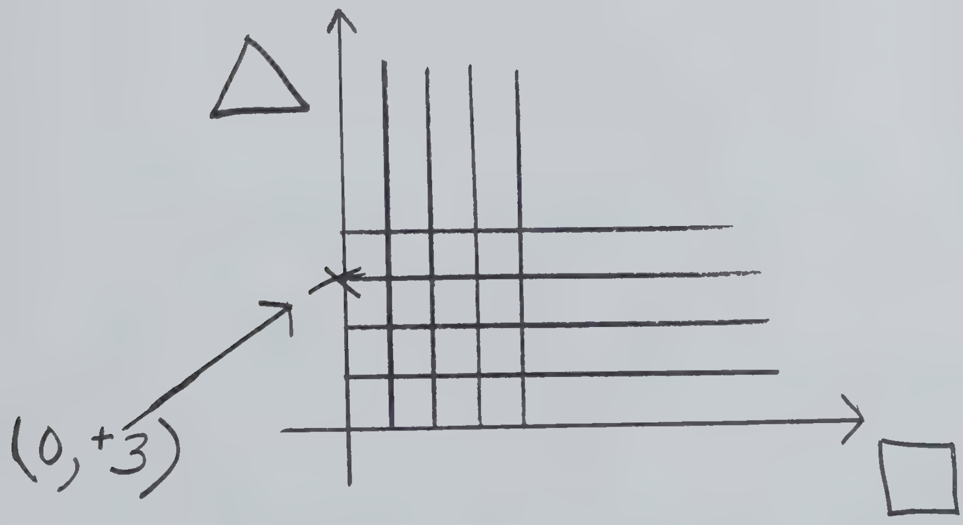
$^{+}1965$ ("positive 1965")

$^{-}3\frac{1}{2}$ ("negative $3\frac{1}{2}$")

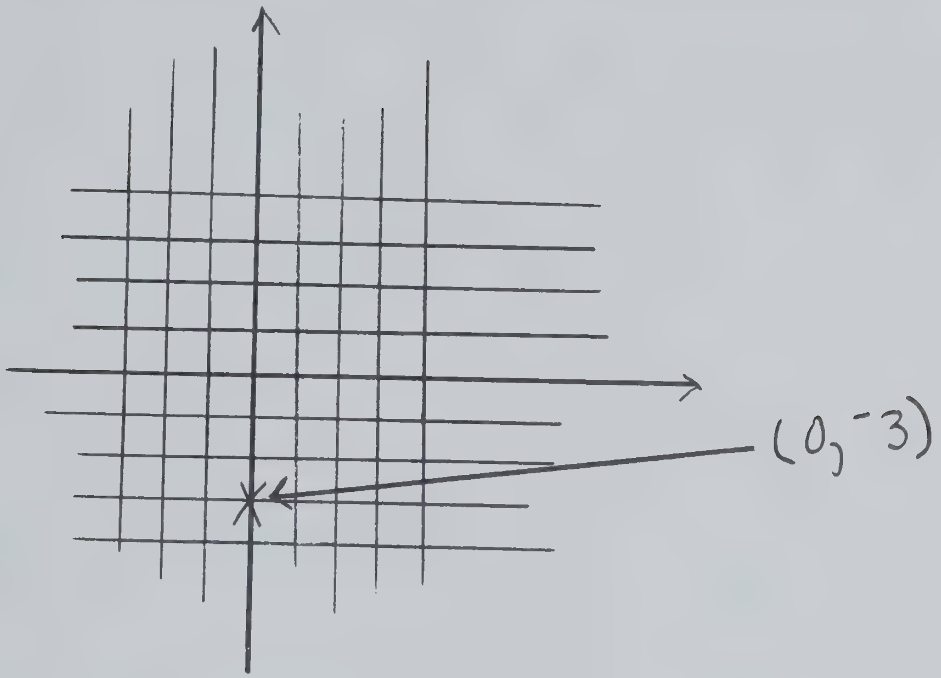
Can you think of any places where such numbers might be useful?

2) Ruth says that numbers with signs could be used on a graph, to distinguish up from down.

For example, $(0, {}^{+}3)$ would be



whereas, $(0, {}^{-}3)$ would be



Do you agree with Ruth?

3) Where would you graph $({}^{+}1, {}^{+}1)$?

4) Where would you graph $({}^{+}1, {}^{-}1)$?

5) Where would you graph $({}^{-}1, {}^{-}1)$?

6) Where would you graph $(^{-}1, ^{+}1)$?

Can you graph each of the following points?

7) $(^{+}3, ^{+}2)$

8) $(^{+}5, ^{+}2)$

9) $(^{+}5, ^{-}2)$

10) $(^{+}6, ^{-}3)$

11) $(^{+}8, ^{-}4)$

12) $(^{-}3, ^{+}1)$

13) $(^{+}5, ^{-}1)$

14) $(^{-}6, ^{-}2)$

8. More Numbers With Signs

In baseball we usually use two numbers to tell the score, for example:

Boston	1
New York	4

If we use numbers with signs we can tell the score with a single number.

Suppose, for example, we agree

that:

Positive numbers mean New York is ahead.

Negative numbers mean Boston is ahead.

1) How would you use one number to give the score New York 4, Boston 1?

2) Suppose the score was $^{+}4$, and suppose a batter for Boston

got on second, scoring a runner from third. What would the score be then?

3) Suppose the next batter for Boston hit a home run with one man on. What would the score be then?

4) The next Boston batter gets on and the following batter hits another homer. What is the

score now?

5) In the next inning, New York is scoreless, but Boston picks up two runs. What is the score now?

9. The Postman Example

We would like to invent an arithmetic for numbers with signs.

To do this, we look at an example.

Suppose a postman brings you a check for \$3. We can represent this as $^+3$. If he brings you a bill for \$2, we can represent that as $^-2$.

1) Suppose the postman brings you a check for \$5, and a bill for \$3. Are you richer or poorer? By how much?

Can you make up a postman story for each problem?

2) $^+2 + {}^+4 = ?$ 3) $^+5 + {}^-2 = ?$

4) $^-2 + {}^-3 = ?$ 5) $^+5 + {}^-6 = ?$

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6) $^{-}7 + ^{+}9 = ?$ 7) $^{-}5 + ^{+}1 = ?$

8) $^{-}3 + 0 = ?$ 9) $^{-}2 + ^{-}5 = ?$

10) $^{+}6 + ^{+}3 = ?$ 11) $^{+}8 + ^{-}1 = ?$

12) $^{-}3 + ^{-}6 = ?$ 13) $^{+}1 + ^{+}12 = ?$

14) $^{+}7 + ^{-}9 = ?$ 15) $^{+}2 + ^{+}17 = ?$

16) Can you make up a story that will correspond to subtraction? How would you explain this?

$$^{+}7 - ^{+}2 = ?$$

Can you make up a postman story for each problem?

17) $^{+}9 - ^{+}2 = ?$ 18) $^{+}15 - ^{+}5 = ?$

19) $^{+}8 - ^{+}2 = ?$ 20) $^{-}5 - ^{-}1 = ?$

21) $^{-}5 + ^{-}1 = ?$ 22) $^{+}10 - ^{+}2 = ?$

23) $^{+}8 - ^{+}7 = ?$ 24) $^{-}5 - ^{-}2 = ?$

25) $^{-}8 - ^{-}1 = ?$ 26) $^{-}8 + ^{-}1 = ?$

27) $^{+}7 - {}^{-}2 = ?$

28) $^{+}7 - {}^{-}4 = ?$

29) $^{+}3 + {}^{+}5 = ?$

30) $^{-}2 + {}^{-}6 = ?$

31) $^{-}9 - {}^{-}3 = ?$

32) $^{-}9 + {}^{+}2 = ?$

33) $^{-}8 - {}^{-}2 = ?$

Can you graph each of these points?

34) $(0, {}^{+}3)$

35) $({}^{+}1, {}^{+}2)$

36) $(0, {}^{-}3)$

37) $({}^{+}2, {}^{-}3)$

38) $({}^{-}3, {}^{-}2)$

39) $({}^{+}5, {}^{+}1)$

40) $({}^{+}5, {}^{-}1)$

41) $({}^{+}7, 0)$

42) $({}^{-}5, {}^{+}3)$

Can you find the truth sets for these open sentences?

43) $6 + \square = 11$

44) $7 + \square = 5$

45) $9 + \square = 6$

46) $9 + \square = 11$

47) $^{+}5 + \square = {}^{+}5$

48) $3 + \square = 1$

10. Equations and Truth Sets

1) The truth set for this equation contains two numbers. Can you find them? (They are both whole numbers.)

$$(\square \times \square) - (13 \times \square) + 22 = 0$$

2) Can you find the truth set for this equation?

$$(\square \times \square) - (15 \times \square) + 44 = 0$$

Can you find the truth set for each of these equations?

3) $(\square \times \square) - (14 \times \square) + 33 = 0$

4) $(\square \times \square) - (8 \times \square) + 15 = 0$

5) $(\square \times \square) - (5 \times \square) + 6 = 0$

6) $(\square \times \square) - (10 \times \square) + 24 = 0$

7) $(\square \times \square) - (10 \times \square) + 21 = 0$

8) $(\square \times \square) - (7 \times \square) + 6 = 0$

9) $(\square \times \square) - (3 \times \square) + 2 = 0$

10) $(\square \times \square) - (21 \times \square) + 108 = 0$

11) $(\square \times \square) - (24 \times \square) + 80 = 0$

12) $(\square \times \square) - (20 \times \square) + 99 = 0$

13) David says he has discovered a shorter way to solve these problems. Have you discovered this "short method" yourself?

Can you find the truth set for each of these equations?

14) $(\square \times \square) - (7 \times \square) + 10 = 0$

15) $(\square \times \square) - (9 \times \square) + 14 = 0$

16) $(\square \times \square) - (25 \times \square) + 144 = 0$

17) $(\square \times \square) - (18 \times \square) + 80 = 0$

18) Have you discovered David's "short method"?

19) Can you solve:

$$(\square \times \square) - (43 \times \square) + 352 = 0 \quad ?$$

11. How Can We Decide Questions in Mathematics?

1) Al and Henry are arguing about the answer to: $\quad {}^{+}2 \times {}^{-}3 = ?$

Al says the answer is $\quad {}^{+}2 \times {}^{-}3 = {}^{-}1.$

Henry says the answer should be: ${}^{+}2 \times {}^{-}3 = {}^{-}5.$

How can they decide who is right?

2) Joe says they can perform an experiment, and settle the matter that way. Do you agree?

3) Brian is skeptical. He asks: How can you make an experiment out of a problem like $\;{}^{+}2 \times {}^{-}3 \overset{?}{=} {}^{-}5$? What do you think?

4) Joe says that chemists do it all the time. What do you think?

5) Brian says that is all well and good for the

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chemists. If you want to Know whether sodium chloride will dissolve in water, you can just dump some sodium chloride into some water, and watch to see what happens.

But what are you going to do about something like $^{+}2 \times {}^{-}3 \stackrel{?}{=} {}^{-}5$?

How would you answer Brian?

Pattern:

One way to settle questions in mathematics is to look for patterns that seem to be important.

6) Is it true that $3 \times 4 = 4 \times 3$?

7) Is it true that $5 \times 7 = 7 \times 5$?

8) Is it true that $1066 \times 1732 = 1732 \times 1066$?

How do you Know? Did you compute?

9) Is it true that $5+1 = 1+5$?

10) Is it true that $3+2 = 2+3$?

11) Is it true that $\frac{31}{61} + \frac{19}{23} = \frac{19}{23} + \frac{31}{61}$?

How do you Know? Are you absolutely sure?

12) Is it true that $\frac{36}{200} + \frac{11}{301} = \frac{11}{301} + \frac{36}{200}$?

Can you decide this without computing?

13) Is $\square \times \triangle = \triangle \times \square$ an identity? Are you absolutely sure?

14) Is $\square + \triangle = \triangle + \square$ an identity? Are you absolutely sure?

15) How does the idea of pattern relate to the preceding questions?

16) Can patterns ever lead you astray?

17) In the system of integers, the number 2 is a "prime" because it has no factors except 1 and itself. Is 3 a prime?

18) Is 4 a prime?

19) Which elements of the following set are primes:

$\{5, 6, 7, 8, 9, 11, 13, 15, 16, 17, 18, 21\}$?

20) Is it true that $2 < 100$?

21) Is it true that $3 < 100$?

22) Is it true that $5 < 100$?

23) Is it true that $11 < 100$?

24) Is it true that $19 < 100$?

25) Is it true that every prime is less than 100?

26) Paul says that what we are calling pattern

is what his father calls "*generalization* *from* experience."

For example, if we find that every summer we can remember was warm, we might *generalize* and say that we believe *next* summer will be warm, and every other summer will also be warm.

Can you give any examples where *you* believe something *because you are generalizing from your experience*?

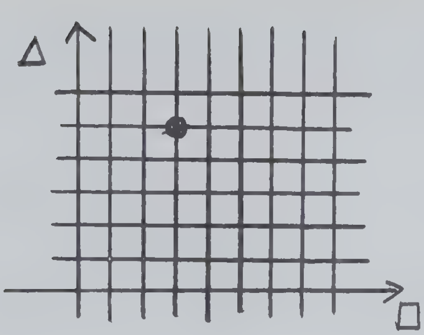
Do you agree that this is what we mean by pattern?

Agreement:

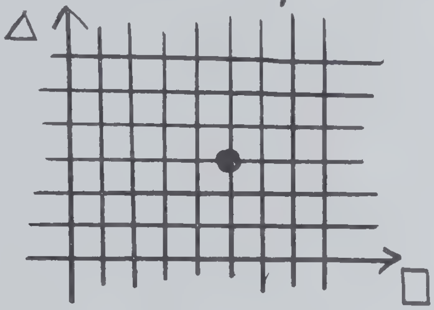
Some statements in mathematics are true because of *agreement*.

27) Is it true that the point $\boxed{3}, \triangle 5$ is

located like this:

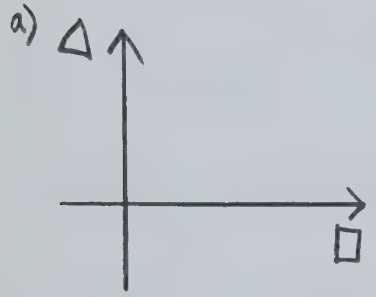


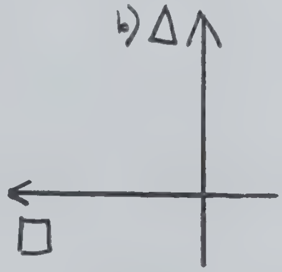
or, should it be graphed this way:

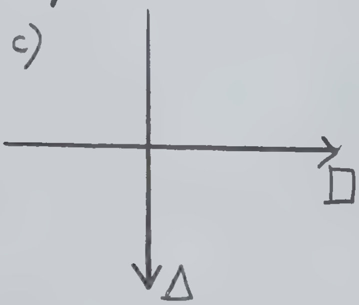


?

Which graph is set up correctly?







Why? Why aren't the others correct?

29) Can you find a statement in mathematics

that you believe because it fits into a pattern?

30) Can you find a statement that is true because of agreement?

31) Can you find a statement in mathematics that you can support by performing an experiment?

32) Can you prove that there is no largest positive integer? Did you use reasoning?

33) Bill says that we can't use reasoning unless we first agree to some rules for "legal" reasoning. Otherwise we'd have to accept bad reasoning as well as good reasoning. Do you agree?

34) Graph $\Delta = \square + 7$.

Find the "pattern" of this graph, and use this pattern to locate some points to the left of the vertical axis.

Can you learn anything from this about the addition of signed numbers?

What is $^{-}1 + {}^{+}7 = ?$

35) What is $^{-}2 + {}^{+}7 = ?$

Can you use the "pattern" on the graph to help you decide?

36) What is $^{-}3 + {}^{+}7$? Can you use the "pattern" on the graph to help you decide?

37) Is it true that $^{-}2 \times {}^{+}6 = {}^{+}6 \times {}^{-}2$?

38) Is it true that $^{+}3$ means the same thing as 3? Why?

39) Is it true that using "wiggly brackets" such as $\{1, 2\}$ means the set consisting of the elements 1 and 2? Why is this true?

7. An Interpretation

1) Brian says that before we can perform an experiment, we have to interpret each symbol in our equation as representing something real. Do you agree?

2) For the equation $^{+}2 + {}^{+}3 = {}^{+}5$, how would you interpret the $^{+}2$? How about the $^{+}3$? How about the +? With your interpretation, does the equation seem to be correct?

3) Wilma suggests this interpretation: The $^{+}2$ will represent \$2 earned, the $^{+}3$ will represent \$3 earned, the + will mean that you earned \$2 and also you earned \$3.

With this interpretation, does the equation seem

to be correct?

4) Jerry says that, for the equation

$$2 \times 3 = 6,$$

if you interpret the 2 as inches and the 3 as inches, then the 6 becomes square inches, and measures area. Do you agree with Jerry?

5) Therefore, says Jerry, if we interpret the equation $2 \times 3 = 6$ to mean \$2 times \$3, the result would be 6 dollars squared, and there isn't any such thing. Do you agree?

6) Can you give an interpretation of

$$2 \times 3 = 6,$$

using money earned or spent?

7) Can you use money earned or spent to

interpret $${}^{-}2 + {}^{-}3 = {}^{-}5 \text{ ?}$$

Does the equation seem to be correct?

8) Ernie says that a better way to interpret

two times something

$$2 \times \underline{\quad\quad} = \underline{\quad\quad}$$

is to say that something happens twice. Do you agree?

9) Can you use Ernie's method to interpret

$$2 \times 3 = 6 \text{ ?}$$

10) Can you use Ernie's method to interpret

$${}^{+}2 \times {}^{+}3 = {}^{+}6 \text{ ?}$$

11) Jane says that

$${}^{+}2 \times {}^{-}3 = {}^{-}6$$

can be interpreted this way:

${}^{-}3$ ↖ a loss of \$3

happened twice ↗ $^{+}2\times$

So, the combination $^{+}2 \times {}^{-}3$

says that on two occasions we lost \$3. The result is that we are poorer ↙

$$^{+}2 \times {}^{-}3 = {}^{-}$$

by an amount of \$6

$$^{+}2 \times {}^{-}3 = {}^{-}6. \nwarrow$$

Do you agree with Jane?

12) Can you use Ernie's idea to interpret

$$^{+}3 \times {}^{-}4 = ?$$

13) Can you use Ernie's idea to interpret

$$^{+}4 \times {}^{-}7 = ?$$

14) Joe says that we can decide

$$^{-}2 \times {}^{+}5 = ?$$

by using the identity

$$\square \times \triangle = \triangle \times \square,$$

to get $^{+}5 \times {}^{-}2 = ?$.

Now, ${}^{-}2$ represents a loss of \$2.

$^{+}5 \times$ means it happened five times

and so, $^{+}5 \times {}^{-}2$

means we lost \$2 five times, so we are poorer

$$^{+}5 \times {}^{-}2 = \underline{\quad}^{\checkmark}$$

by an amount of \$10.

$$^{+}5 \times {}^{-}2 = {}^{-}\underset{\nwarrow}{10}$$

Do you agree?

15) Can you represent

$$^{-}2 \times {}^{-}3 = ?$$

13. Arithmetic for Signed Numbers – Continued

1. If $^{+}2 \times {}^{-}3$ means a three dollar loss that happened twice, can you make up a story for $^{-}2 \times {}^{-}3$, that would mean a three dollar loss "unhappening" twice.

2. Jerry says a profit can happen twice ($^{+}2 \times$ ___) if we make two sales. It can "unhappen" twice if we find that yesterday we accepted two bad checks. Do you agree with Jerry?

3. Can you make up a story for $^{-}2 \times {}^{+}3$, that will mean a \$3 profit "unhappening" twice?

4. Can you make up a story for $^{+}2 \times {}^{+}3$?

5. Can you make up a story where a \$3 loss will "unhappen" twice? How can a loss "unhappen"?

6. Beth says that you get richer if a profit happens, or if a loss "unhappens." Do you agree?

7. Steve says you get poorer if a loss happens, or if a profit "unhappens." Do you agree with Steve?

8. As a businessman, which would you like, and which would you dislike?

a) You make a sale (a profit happens)

b) An article is damaged before sale (a loss happens)

c) A check which you had accepted turns out to be a bad check (a profit "unhappens")

d) Some bad checks are made good (a loss "unhappens")

9) Jerry says that

$$^-2 \times {}^+3$$

means a profit of \$3 "unhappens" twice. Are you richer or poorer? What answer should we write for $^-2 \times {}^+3$?

10) Allen says there is another way to decide $^-2 \times {}^+3$. We have the identity

$$\square \times \triangle = \triangle \times \square.$$

Consequently, we can say

$$^-2 \times {}^+3 = {}^+3 \times {}^-2.$$

Now, $^+3 \times {}^-2$ means a loss of \$2

$$\underline{\qquad} \quad {}^-2$$

which happens three times

$$^+3 \times \underline{\qquad}.$$

Consequently, $^{+}3 \times {}^{-}2 = {}^{-}6$.

Do you agree with Allen?

11) The identity $\square \times \triangle = \triangle \times \square$ is so important that it has been given a name. It is called the commutative law for multiplication.

Can you use the commutative law for multiplication to decide the answer for $^{-}3 \times {}^{+}4$?

12) Can you make up a story for

$$^{-}3 \times {}^{+}4$$

using the idea of something "unhappening"?

13) Lou says that he can use another pattern to decide $^{+}2 \times {}^{-}1$.

He says he begins by making a graph for the open sentence $\triangle = 1 + (2 \times \square)$.

Can you figure out how Lou's method works?

14. How many ways do you know to decide on an answer for

$$^{+}2 \times {}^{-}5 = \_\_\_ ?$$

15. What answer should we write for

$$^{-}3 \times {}^{-}5 = \_\_\_ ?$$

14. More Identities

1. What do we mean by saying that an open sentence is an identity?

Which of these are identities?

2. $\triangle \times \square = \square \times \triangle$

3. $\square + \triangle = \triangle + \square$

4. $(\square \times \triangle) + \square = (\square \times \square) + \triangle$

5. $\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$

6. $(\square \times \triangle) + \square = \square + (\square \times \triangle)$

7. $(\square \times \triangle) + \square = \square + (\triangle \times \square)$

8. $\square \times [(\square \times \square) + 2] = 3 \times (\square \times \square)$

9. $(\square \times \square) \times [(\square \times \square) + 11] = (6 \times \square) \times [(\square \times \square) + 1]$

10. $(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + [2 \times (\square \times \triangle)] + (\triangle + \triangle)$

11. $(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$

12) $(\square+4)\times(\square+4)=(\square\times\square)+(8\times\square)+16$

13) $(\square+1)\times(\square+1)=(\square\times\square)+(2\times\square)+1$

14) $\square\times(\triangle+5)=(\square\times\triangle)+(\square\times5)$

15) $\square\times(\triangle\times\nabla)=(\square\times\triangle)\times\nabla$

16) $\square+(\triangle+\nabla)=(\square+\triangle)+\nabla$

17) $(\square\times\triangle)+\nabla=\square\times(\triangle+\nabla)$

18) $\square\times(\triangle+3)=(\square\times\triangle)+(\square\times3)$

19) Write down your cumulative list of identities. Can you discover anything about this list?

20) Can you make up five new identities which are not on the cumulative list?

21) Is this an identity?

$$1\times\square=\square$$

22) Is this an identity? $\square+0=\square$

5. Implication

1) Hal says that the list of identities will soon be too long. How can we work with it more effectively?

2) Lou says we can cross out certain identities, because they follow from others. Can you find some identity on the list which can be crossed out, since it is a consequent of some other identity?

3) Jerry says the identity

$$\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$$

can be crossed out. Jerry's argument goes like this:

We know that $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$ is an identity. We know we can put any number in the ∇ so long as we put it in the ∇ every time the

placeholder ∇ appears. Now, we put 3 in the ∇. The result is:

$$\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$$

Consequently, we don't need this last identity on our list — if we cross it out, we can get it back any time we want by using the identity:

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla).$$

4) Can you make up five more identities to add to the list?

5) Can you cross any of them off of the list the way Jerry did?

6) Can you cross any more identities off the list by using Jerry's method?

7) Ellen says that what we really need here is

the idea of implication. She gives this example of a list of statements:

1) All Bill's marbles are blue.

2) Joe won five marbles from Bill.

3) Joe won five blue marbles.

Ellen says she can cross off the third statement from the list. The first two statements imply the third. [We write this $1,2 \Rightarrow 3$.]

What do you think of Ellen's idea?

8) Can you make up a list of three statements where the first two statements imply the third, but we need both of the first two statements in order to get the third statement?

9) Joe suggests this list for question 8:

1) My cousin plays baseball.

2) My cousin is in the 6th grade.

3) My cousin is a very good baseball player

Is Joe's answer correct?

10) Beth says that Joe's list does not work. She says the third sentence tells you something new that is not implied by the first two. Do you agree with Beth?

11) Nancy suggested this list:

1) All the teachers in this school are tall.

2) This school is the largest school in Oregon.

3) The teacher in this class is tall.

Does Nancy's list answer question 8?

12) Jerry says Nancy's list does not answer

question 8, because $1 \Rightarrow 3$, and you don't need 2 in order to get 3. Do you agree with Jerry?

13) Can you make up a list to answer question 8?

14) Al suggested this list:

1) All policemen are tall.

2) Henry is a policeman.

3) Henry is tall.

Does Al's list answer question 8?

15) Joe says we could write the implication in Al's list this way: $1, 2 \Rightarrow 3$

Is Joe right?

16) Jerry says if you tell someone Al's first two statements, then he would know the third statement even if you didn't tell him.

Is this what Ellen meant by "implication"?

17) Can you make any more lists to answer question 8?

18) Bart suggested this list:

1) Jerry had 5 marbles.

2) Jerry lost 2 marbles during the day.

3) At the end of the day, Jerry had 3 marbles.

Does Barts list answer question 8?

19) Al suggested this list:

1) There are twenty-five students in this class.

2) Seventeen of the students in this class are boys.

3) In this class, $\frac{17}{25}$ of the students are boys.

Do you think Al's list is an acceptable answer to question 8?

20) Woody says that sometimes you can get several

implications from the first two statements. As an example, Woody suggested this:

1) There are twenty-five students in this class
2) Seventeen of the students in this class are boys.
3) In this class, $\frac{17}{25}$ of the students are boys.
4) There are eight girls in this class.
5) In this class, $\frac{8}{25}$ of the students are girls.
6) There are more boys than girls in this class.

"Now," says Woody, "if we have this list of six statements, we can cross off the 3rd because $1,2 \Rightarrow 3$. But we can cross off the 4th because $1,2 \Rightarrow 4$. And we can cross off the 5th because $1,2 \Rightarrow 5$."

Do you agree with Woody?

21) On Woody's list, can you cross off the 6th statement?

Is it true that $1,2 \Rightarrow 6$?

22) Ellen says that Woody's list is a perfect example of what she meant when she said that a list of statements can sometimes be shortened by using the idea of implication.

Do you agree with Ellen?

23) Wayne said that if you had this list of identities:

1) $\square + \triangle = \triangle + \square$

2) $(3 \times \triangledown) + \triangle = \triangle + (3 \times \triangledown)$

then you could cross off the second identity, because $1 \Rightarrow 2$.

Do you agree?

24) John said that he doesn't think Wayne is right.

What do you think?

25) Wayne explained his idea this way:

The identity $\square + \triangle = \triangle + \square$ tells me that, whenever

I'm going to add two numbers, it doesn't matter which one I write on the left, and which one I write on the right.

Now, I want to add $(3 \times \nabla)$ and Δ. From this first identity I know that I can write the $(3 \times \nabla)$ on the left $(3 \times \nabla) + \Delta$

or I can write it on the right $\Delta + (3 \times \nabla)$

Either way, I get the same answer:

$$(3 \times \nabla) + \Delta = \Delta + (3 \times \nabla).$$

Do you agree with Wayne?

26) Paul says that Wayne's argument could be written like this:

$$\square + \Delta = \Delta + \square \quad \Longrightarrow \quad (3 \times \nabla) + \Delta = \Delta + (3 \times \nabla)$$

Is Paul right?

27) Can you make up five identities which follow from the identity $\square + \triangle = \triangle + \square$?

28) Beth suggested this identity in answer to question 27:

$$(\square \times 3) + (5 \times \triangle) = (5 \times \triangle) + (\square \times 3)$$

Is Beth right?

29) Joanne suggested this identity in answer to question 27:

$$(\square \times 3) + (5 \times \triangle) = (5 \times \triangle) + (3 \times \square).$$

Is Joanne right?

30) Paul says that Joanne's identity does not follow from $\square + \triangle = \triangle + \square$ alone. He says that you need both $\square + \triangle = \triangle + \square$ and $\square \times \triangle = \triangle \times \square$ in order to get Joanne's identity. Is Paul right?

31) Jerry says we could write:

$$\left.\begin{array}{l} \square+\triangle=\triangle+\square \\ \text{and} \\ \qquad \square\times\triangle=\triangle\times\square \end{array}\right\} \Longrightarrow \quad (\square\times 3)+(5\times\triangle)=(5\times\triangle)+(3\times\square)$$

Do you agree?

32) Paul says that the identity $\square+\triangle=\triangle+\square$ permits you to write

$$(\square\times 3)+(5\times\triangle)=(5\times\triangle)+(\square\times 3)$$

and then, the identity $\square\times\triangle=\triangle\times\square$ permits us to write

$$(\square\times 3)+(5\times\triangle)=(5\times\triangle)+(3\times\square).$$

Do you agree with Paul?

33) Henry says you can take an identity like

$$\square+\triangle=\triangle+\square$$

and write whatever you want in the 1st $\square$, provided

you write the same thing in the 2nd $\square$.

Is Henry right?

34) Using his method, Henry writes $\nabla + 7$ in both squares, and gets

$$(\nabla + 7) + \Delta = \Delta + (\nabla + 7).$$

Is Henry right? Is this an identity?

35) Gail says we could write

$$\square + \Delta = \Delta + \square \Longrightarrow (\nabla + 7) + \Delta = \Delta + (\nabla + 7).$$

Is Gail right?

36) Can you make up an identity which follows from $\square + \Delta = \Delta + \square$, by using Henry's method?

1.C. "Adding" Statements

1) Mary likes to write implication this way:

Cats have four legs.
Propyl is a cat.
Propyl has four legs.

How do you like Mary's idea?

2) Can you make up an example such as Mary's, where two statements together imply a third?

3) Bruce gave this example:

My father is a lawyer.
Jerry's father is in the same profession as my father.
Jerry's father is a lawyer.

Is this example correct?

4) Evelyn gave this example:

Silver is a metal.
My spoon is made of stainless steel.
My spoon is made of metal.

Is this a good example of two statements that imply a third statement?

5) Paul disagrees with Evelyn's example. He says that if you *tell* somebody that "Silver is a metal," and if you *tell* them "My spoon is made of stainless steel," you have not *told* them that your spoon "is made of metal."

Paul agrees that they may *Know* that stainless steel is a metal, but you didn't *tell* them.

He says that *implication* deals with what you *tell* them, and not with what they *know*.

Do you agree with Paul?

6) Gene suggested this example:

1) All trees have leaves.
2) The catalpa tree has green leaves.

---

3) Maple trees have leaves.

Is this a case where $1,2 \Longrightarrow 3$?

7) Jerry says Gene's example is not a case where $1,2 \Longrightarrow 3$. Do you agree? How would you write the implication?

8) Kathy says she would write Gene's example this way:

$$1 \Longrightarrow 3.$$

Do you agree?

9) Marie gave this example:

I have two American coins in my hand.
Their total value is eleven cents.

One coin is a dime and the other is a penny.

What do you think of this example?

10) Can you make up an example of two statements

which *together* imply a third statements? [That is, if you *tell* someone the first two statements, you have, in effect, automatically *told* them the third.]

11) Is this a correct implication;

$$\begin{array}{c} \square + \triangle = \triangle + \square \\ \square \times \triangle = \triangle \times \square \\ \hline (3\times\square)+(\triangledown\times 2) = (2\times\triangledown)+(3\times\square) \end{array} \quad ?$$

Which of the following are correct statements of *implication*?

12)
$$\begin{array}{c} \square \times (\triangle + \triangledown) = (\square\times\triangle)+(\square\times\triangledown) \\ (\square + \triangle) = (\triangle + \square) \\ \hline 3\times(\triangle + 5) = (3\times 5)+(3\times\triangle) \end{array}$$

13)
$$\begin{array}{c} \square + \triangle = \triangle + \square \\ \triangle \times \square = \square \times \triangle \\ \hline (3\times\square) = (2\times\square)+\square \end{array}$$

14)
$$\begin{array}{c} \square \times (\triangle + \triangledown) = (\square\times\triangle)+(\square\times\triangledown) \\ \square \times \triangle = \triangle \times \square \\ \hline \square \times (\triangle + \triangledown) = (\triangle\times\square)+(\triangledown\times\square) \end{array}$$

15) $\square\times(\triangle+\triangledown)=(\square\times\triangle)+(\square\times\triangledown) \Rightarrow \square\times(\triangle+2)=(\square\times\triangle)+(\square\times 2)$

16) Paul wants to find the smallest list of identities that will imply all the others.

Can you begin such a list?

17) Paul's father says that some identities are so important they have been given names.

Can you name any of the identities?

18) According to Paul's father, these are some of the identities that have names;

$\square + \triangle = \triangle + \square$	commutative law for addition
$\square \times \triangle = \triangle + \square$	commutative law for multiplication
$(\square + \triangle) + \triangledown = \square + (\triangle + \triangledown)$	associative law for addition
$(\square \times \triangle) \times \triangledown = \square \times (\triangle \times \triangledown)$	associative law for multiplication
$\square \times (\triangle + \triangledown) = (\square \times \triangle) + (\square \times \triangledown)$	distributive law
$\square \times 1 = \square$	multiplication identity law

$\square + 0 = \square$ additive identity law

Is Paul's father right?

19) Jerry says that the law

$$\square \times 1 = \square$$

serves to determine what we mean by "1". He says we can explain what we mean by "2" by agreeing that "2" is just a new name for $1+1$.

He says "3" is just a new name for

$$2+1.$$

Can you explain "4"? Can you explain what we mean by "5"?

11. Equations, Identities, Graphs, and Signed Numbers

1) Paul says the major topics in this course are:

Equations (and Truth Sets)

Identities

Graphs

Signed Numbers

Do you agree with Paul's list?

Can you find the truth set for each open sentence?

2) $3+\Box=5$

3) $1+(2\times\Box)=13$

4) $3+(2\times\Box)=21$

5) $1+(2\times\Box)=15$

6) $1+(2\times\Box)=6$

7) $1+(2\times\Box)=8$

8) $2+(3\times\Box)=6$

9) $1+(11\times\Box)=26$

10) ________ sits in the front row.

11) ________ is the teacher of this class.

12) $7 + \square = 11$

13) $1 < 1 + \square < 8$ (Use only whole numbers.)

14) $3 < 1 + (2 \times \square) < 11$ (Use only whole numbers.)

15) $3 < 1 + (2 \times \square) < 11$ (Use whole numbers and fractions.)

16) $(\square \times \square) - (13 \times \square) + 22 = 0$

17) $(\square \times \square) - (5 \times \square) + 6 = 0$

18) $(\square \times \square) - (15 \times \square) + 50 = 0$

19) $(\square \times \square) - (8 \times \square) + 15 = 0$

20) $(\square \times \square) - (9 \times \square) + 14 = 0$

Which of the following are identities:

21) $\square \times [(\square \times \square) + 26] = [9 \times (\square \times \square)] + 24$

22) $$[(\square\times\square)\times(\square\times\square)]+[71\times(\square\times\square)]+120$$
$$=\square\times[\{14\times(\square\times\square)\}+154]$$

23) $(\square+3)\times(\square+5)=(\square\times\square)+(8\times\square)+15$

24) $(\square+2)\times(\square+5)=(\square\times\square)+(7\times\square)+10$

25) $(\square+3)\times(\square+4)=(\square\times\square)+(7\times\square)+12$

26) $(\square+2)\times(\square+11)=(\square\times\square)+(13\times\square)+22$

27) $(\square+3)\times(\square+6)=(\square\times\square)+(8\times\square)+15$

28) $(\square+2)\times(\square+8)=(\square\times\square)+(10\times\square)+16$

29) $(\square+2)\times(\square+9)=(\square\times\square)+(5\times\square)+14$

Can you fill in the blank spaces so as to get an identity?

30) $(\square+2)\times(\square+9)=(\square\times\square)+(\overset{\swarrow}{\underline{\quad}}\times\square)+\overset{\swarrow}{\underline{\qquad}}$

31) $(\square+3)\times(\square+6)=(\square\times\square)+(\overset{\swarrow}{\underline{\quad}}\times\square)+\overset{\swarrow}{\underline{\qquad}}$

32) $(\square+3)\times(\square+11)=(\square\times\square)+(\overset{\swarrow}{\underline{\quad}}\times\square)+\overset{\swarrow}{\underline{\qquad}}$

33) $(\square+5)\times(\square+7)=(\square\times\square)+(\overset{\swarrow}{\underline{\qquad}}\times\square)+\overset{\swarrow}{\underline{\qquad}}$

34) $(\square+2)\times(\square+6)=(\square\times\square)+(\overset{\swarrow}{\underline{\qquad}}\times\square)+\overset{\swarrow}{\underline{\qquad}}$

Can you complete each identity?

35) $(\square+5)\times(\square+6)=$ ________________

36) $(\square+5)\times(\square+11)=$ ________________

37) $(\square+2)\times(\square+13)=$ ________________

38) $(\square+1)\times(\square+2)=$ ________________

39) David says that the identity

$$\square\times(\triangle+\triangledown)=(\square\times\triangle)+(\square\times\triangledown)$$

gives him two different ways to multiply

$$4\times2\tfrac{1}{4}.$$

Can you figure out David's two methods?

Multiply:

40) $5\times2\tfrac{1}{5}$

41) $6\times5\tfrac{1}{3}$

42) $8 \times 3\frac{1}{2}$

43) $9 \times 2\frac{2}{9}$

44) $2 \times 2\frac{1}{2}$

45) $3 \times 5\frac{2}{3}$

46) $10 \times 3\frac{1}{2}$

47) $4 \times 3\frac{1}{4}$

Graph, using both negative and positive numbers:

48) $\triangle = 1 + (3 \times \square)$

49) $\triangle = 3 + (4 \times \square)$

50) $\triangle = 1 + (2 \times \square)$

51) $\triangle = (3 \times \square) - 2$

52) $\triangle = (5 \times \square) - 4$

53) $\square + \triangle = 20$

54) $\square \times \triangle = 36$

55) $(\square \times \square) + (\triangle \times \triangle) = 25$

56) $\square + \triangle = 12$

57) $\square + \triangle = 25$

58) $\square + \triangle = 8$

Can you find the truth set for each open sentence?

59) $7 + \square = 5$

60) $8 + \square = 4$

61) $15 + \square = 10$

62) $^{+}12 + \square = {}^{+}8$

63) $^{+}1 + \square = {}^{-}1$

Can you decide what each answer should be?

Explain with a story.

64) ${}^{+}2 + {}^{+}3 =$ ___

65) ${}^{+}2 + {}^{-}1 =$ ___

66) ${}^{+}2 + {}^{-}3 =$ ___

67) ${}^{+}5 - {}^{+}2 =$ ___

68) ${}^{+}5 - {}^{-}1 =$ ___

69) ${}^{+}7 - {}^{+}2 =$ ___

70) ${}^{+}7 - {}^{-}1 =$ ___

71) ${}^{-}2 + {}^{-}3 =$ ___

72) ${}^{+}2 \times {}^{+}3 =$ ___

73) ${}^{+}2 \times {}^{-}3 =$ ___

74) ${}^{-}2 \times {}^{+}5 =$ ___

75) ${}^{-}2 \times {}^{-}3 =$ ___

76) ${}^{-}5 \times {}^{-}4 =$ ___

77) ${}^{+}5 \times {}^{-}3 =$ ___

78) ${}^{+}2 \times {}^{-}11 =$ ___

18. Derivations

1) Brian says this list can be shortened:

List #I 1) $\square\times\triangle=\triangle\times\square$

2) $\square\times(\triangle+\nabla)=(\square\times\triangle)+(\square\times\nabla)$

3) $\square\times(\triangle+\nabla)=(\triangle\times\square)+(\square\times\nabla)$

What do you think? Can you shorten the list?

2) Henry says that 1) and 2) $\Longrightarrow$ 3. Do you agree?

3) Nancy says that you can start with 2):

$$\square\times(\triangle+\nabla)=(\square\times\triangle)+(\square\times\nabla)$$

"Now," she says, "number 1) says we can flip the square and the triangle:

$$\square\times(\triangle+\nabla)=(\underset{\curvearrowleft}{\overset{\curvearrowright}{\square\times\triangle}})+(\square\times\nabla)$$

and get $\square\times(\triangle+\nabla)=(\triangle\times\square)+(\square\times\nabla)$."

Is Nancy right?

4) Paul calls this "flipping on the multiplication sign

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)."$$

Does Paul's phrase sound descriptive?

5) Can this list be shortened?

List #II 1) $\square \times \triangle = \triangle \times \square$

2) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$

3) $\square + \triangle = \triangle + \square$

4) $\square \times (\triangle + \nabla) = (\nabla \times \square) + (\triangle \times \square)$

6) Can you start with #2) and fill in the missing steps until you end up with #4)?

2) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$

____

____

____

____

4) $\square \times (\triangle + \nabla) = (\nabla \times \square) + (\triangle \times \square)$

The idea is to take just one step each time.

7) Is this derivation correct?

2) $\square \times (\triangle + \triangledown) = (\square \times \triangle) + (\square \times \triangledown)$

i) $\square \times (\triangle + \triangledown) = (\triangle \times \square) + (\square \times \triangledown)$

ii) $\square \times (\triangle + \triangledown) = (\triangle \times \square) + (\triangledown \times \square)$

4) iii) $\square \times (\triangle + \triangledown) = (\triangledown \times \square) + (\triangle \times \square)$

8) How can you justify the step from 2) to i)?

9) How can you justify the step from i) to ii)?

10) How can you justify the step from ii) to iii)?

11) Did we end up where we were supposed to?

12) Wayne says you can justify the step from 2) to i) by identity 1) on the list, which permits you to flip on the multiplication sign

$\square \times (\triangle + \triangledown) = (\underset{\uparrow}{\square} \times \triangle) + (\square \times \triangledown)$ Do you agree?

13) Nancy says you can justify the step from i) to ii) by using identity 1) on the list, which lets you flip on this multiplication

$\square \times (\triangle + \triangledown) = (\triangle \times \square) + (\square \underset{\uparrow}{\times} \triangledown)$ Do you agree?

14) Jerry says you can justify the step from ii) to iii) by using identity #3) on list #II, which lets you interchange this number

$$\square \times (\triangle + \triangledown) = \boxed{(\triangle \times \square)} + (\triangledown \times \square)$$

and this number

$$\square \times (\triangle + \triangledown) = (\triangle \times \square) + \boxed{(\triangledown \times \square)}$$

by flipping on this addition sign

$$\square \times (\triangle + \triangledown) = (\triangle \times \square) \underset{\uparrow}{+} (\triangledown \times \square)$$

Do you agree?

15) In question 14), did Jerry use the identity

$$\square + \triangle = \triangle + \square$$

correctly, or not?

If these were too easy, now we'll give you a hard one:

List # III 1) $\square \times \triangle = \triangle \times \square$

2) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$

3) $\square + \triangle = \triangle + \square$

4) $(a \times b) + (c \times d) = (a \times c) + (a \times d) + (b \times c) + (b \times d)$

16) Can List # III be shortened?

17) Charlie wants to shorten List # III. He claims that 1), 2), and 3) $\Longrightarrow$ 4). Is his description correct?

i) $(a+b) \times (c+d) = (a+b) \times (c+d)$

ii) $(a+b) \times (c+d) = [(a+b) \times c] + [(a+b) \times d]$

iii) $(a+b)\times(c+d)=[c\times(a+b)]+[(a+b)\times d]$

iv) $(a+b)\times(c+d)=[c\times(a+b)]+[d\times(a+b)]$

v) $(a+b)\times(c+d)=(c\times a)+(c\times b)+[d\times(a+b)]$

vi) $(a+b)\times(c+d)=(a\times c)+(c\times b)+[d\times(a+b)]$

vii) $(a+b)\times(c+d)=(a\times c)+(b\times c)+[d\times(a+b)]$

viii) $(a+b)\times(c+d)=(a\times c)+(b\times c)+(d\times a)+(d\times b)$

ix) $(a+b)\times(c+d)=(a\times c)+(b\times c)+(a\times d)+(d\times b)$

x) $(a+b)\times(c+d)=(a\times c)+(a\times d)+(b\times c)+(d\times b)$

xi) $(a+b)\times(c+d)=(a\times c)+(a\times d)+(b\times c)+(b\times d)$

18) Mary says it is always legal to start with a "trivial" identity, such as

$$(a+b)\times(c+d)=(a+b)\times(c+d).$$

[A "trivial" identity has exactly the same thing on both sides.] Do you agree?

19) Can you justify each step in # 17)?

20) Larry says he can justify the step from i) to ii) by using the distributive law:

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla).$$

Larry says we can substitute any number into the 1st square, as long as we put it also into the two other squares.

He puts a+b into the square, c into the $\triangle$, and d into the ∇.

$$\boxed{a+b} \times (\triangle\!\!\!\!\!\:c + \nabla\!\!\!\!\!\:d) = \cdot \underset{?}{\overset{\cdot}{\uparrow}} \cdot$$

What "right-hand" side does Larry get?

21) Does Larry succeed in getting from step i) to step ii)?

22) Can this list be shortened?

List # IV 1) $\square \times \triangle = \triangle \times \square$

2) $\Box \times (\Delta + \nabla) = (\Box \times \Delta) + (\Box \times \nabla)$

3) $\Box + \Delta = \Delta + \Box$

4) $a + (b \times c) = (c \times b) + a$

23) Marie says that 1), 2), and 3) $\Longrightarrow$ 4). Do you think she is right?

24) Alan says that we can derive 4), if we start with the "trivial" identity

$$a + (b \times c) = a + (b \times c).$$

Can you complete the derivation?

19. Generalizations

1) Ruth's list of identities includes these:

$$3 \times \square = \square \times 3$$

$$4 \times \square = \square \times 4$$

$$7 \times \square = \square \times 7$$

$$1960 \times \square = \square \times 1960$$

$$1{,}000{,}000 \times \square = \square \times 1{,}000{,}000$$

$$71 \times \square = \square \times 71$$

Can you find a way to shorten Ruth's list?

2) Jerry's list of identities includes these:

$$\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$$

$$\square \times (\triangle + 5) = (\square \times \triangle) + (\square \times 5)$$

$$\square \times (\triangle + 2) = (\square \times \triangle) + (\square \times 2)$$

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Can you find a way to shorten Jerry's list?

3) Can you make up some more identities like those on Ruth's list?

4) David says that on Ruth's list you can take any number and write it here

$$\overset{\downarrow}{\underline{\quad\quad}} \times \square = \square \times \underline{\quad\quad}$$

as long as you also write the same number here

$$\underline{\quad\quad} \times \square = \square \times \overset{\downarrow}{\underline{\quad\quad}}$$

Do you agree?

5) Marie says we can write this

$$\text{"}\underline{\quad\quad}\text{"} \times \square = \square \times \text{"}\underline{\quad\quad}\text{"}$$

Do you agree?

6) Can you explain what Marie means by writing "___"?

7) Alan says we could also write this

$$\triangle \times \square = \square \times \triangle$$

Do you agree

8) Can you make up some more identities like those on Jerry's list (in question 2)?

9) Nancy's list of identities includes these:

$$\square + 3 = 3 + \square$$

$$\square + 8 = 8 + \square$$

$$\square + 11 = 11 + \square$$

$$\square + 500 = 500 + \square$$

$$\square + 1492 = 1492 + \square$$

Can you find a way to shorten Nancy's list?

10) Lucille's list includes these identities:

$$(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$$

$(\square+5)\times(\square+5)=(\square\times\square)+(5\times\square)+25$

$(\square+4)\times(\square+4)=(\square\times\square)+(8\times\square)+16$

Can you make up some more identities of this form?

11) Can you find a way to shorten Lucille's list?

12) Can you complete these identities?

a) $(\square+2)\times(\square+2)=(\square\times\square)+$ ____?____

b) $(\square+5)\times(\square+6)=(\square\times\square)+$ ____?____

c) $(\square+3)\times(\square+7)=(\square\times\square)+$ ____?____

d) $(\square+2)\times(\square+5)=(\square\times\square)+$ ____?____

13) Can you use the identity

$(a+b)\times(c+d)=(a\times c)+(a\times d)+(b\times c)+(b\times d)$

to help you in question 12?

14) Can you use question 13 to help shorten *your* list of identities?

15) Can you makeup an identity that will generalize this pattern?

$$\frac{1}{2} + \frac{1}{3} = \frac{3+2}{2 \times 3}$$

$$\frac{1}{4} + \frac{1}{7} = \frac{7+4}{4 \times 7}$$

$$\frac{1}{4} + \frac{1}{5} = \frac{5+4}{4 \times 5}$$

16) Jerry's list includes these identities:

$$\frac{1}{\square} + \frac{1}{2} = \frac{2+\square}{2 \times \square}$$

$$\frac{1}{\square} + \frac{1}{3} = \frac{3+\square}{3 \times \square}$$

$$\frac{1}{\square} + \frac{1}{4} = \frac{4+\square}{4 \times \square}$$

Can you make up some more identities of this same form ?

17) Can you find a way to shorten Jerry's List (in problem 16) ?

18) If we keep shortening our List in this way, what do you think the final result will be? What will be the shortest List that will still be "complete" ?

20. Repeated Applications

1) Mary's List of identities includes these:

$$\triangle + \square + \nabla = \nabla + \square + \triangle$$

$$\square + 2 + \nabla + 3 = 3 + \nabla + 2 + \square$$

$$\triangle + 5 + \square + 9 + \nabla = \nabla + 9 + \square + 5 + \triangle$$

Can you find a way to shorten Mary's List?

2) Jerry says that

$$\square + \triangle = \triangle + \square$$

implies

$$\square + 2 + \nabla + 3 = 3 + \nabla + 2 + \square.$$

Do you agree?

3) Jerry worked out the implication this way:

i) $\square + 2 + \triangledown + 3 = \square + 2 + \triangledown + 3$

(He started with a "trivial" identity.)

ii) $\square + 2 + \triangledown + 3 = 2 + \square + \triangledown + 3$

iii) $\square + 2 + \triangledown + 3 = 2 + \triangledown + \square + 3$

iv) $\square + 2 + \triangledown + 3 = \triangledown + 2 + \square + 3$

v) $\square + 2 + \triangledown + 3 = \triangledown + 2 + 3 + \square$

vi) $\square + 2 + \triangledown + 3 = \triangledown + 3 + 2 + \square$

vii) $\square + 2 + \triangledown + 3 = 3 + \triangledown + 2 + \square$

Is Jerry's demonstration correct?

4) Can you explain how Jerry got from i) to ii)?

5) Can you explain how Jerry got from ii) to iii)?

6) Can you explain each step in Jerry's demonstration?

7) Can you shorten Mary's List

in question 1)?

8) What do you suppose is the shortest "complete" List of identities?

21. Names for Numbers

1) (2+5) is a name for some number. Can you give another name for this same number?

2) How many names can you give for the number (7+8)?

3) Jerry says that the meaning of the sign "=" is that you have two names for the same number. For example,

$$\frac{2}{3} \times \frac{3}{4} = \frac{1}{4} + \frac{1}{4}$$

really means that

$$\frac{2}{3} \times \frac{3}{4}$$

is a name for some number, and

$$\frac{1}{4} + \frac{1}{4}$$

is a name *for this same number*.

4) If we had a close-up photo of Pablo Casals, and a distance shot of him, and a profile shot, and a picture from the rear, we would have photos of how many people?

5) If we have these names

$7+5$

$18-6$

$18 \times \frac{2}{3}$

$\frac{1}{4} \times 20$

$5+8$

$11+7$

$21-9$

2×6,

how many different numbers do we have?

6) Joe says that

$\frac{1}{3} \times \frac{3}{8}$

$2 \times \frac{1}{16}$

$1 - \frac{7}{8}$

are all names for the same number.

Do you agree?

7) Mary says that

$\frac{1}{2} + \frac{1}{2}$

$2 - 1$

$7 \times \frac{1}{7}$

$21 \times \frac{1}{21}$

$\frac{1}{4} \overset{1}{+} \frac{3}{4}$

$\frac{7}{8} + (2 \times \frac{1}{16})$

are all names for the same number.

Do you agree?

8) How many different numbers do we have on this List?

$3 + 2$

$7 \times \frac{5}{7}$

$\frac{5}{3} \times 3$

$\frac{21}{8} \times 8$

3×7

5

$28 - 2$

$25 - 4$

${}^{+}22 + {}^{-}1$

${}^{+}20 - {}^{-}1$

9) In question 8, which number is represented on the list by the most names? How many names for this number are there on the list?

10) Jerry says that we may always replace one name for a number by any other name for that same number. Do you agree?

11) "Elizabeth Wilson sits in the back row." Can you replace "Elizabeth Wilson" by some other name for the same girl?

12) Marie says that the identity

$$\triangle + \square = \square + \triangle$$

really means that

$$3+4$$

is a name for a number, and

$$4+3$$

is another name for that same number.

Do you agree?

13) Al says that

$$a + (b+c) = a + (c+b)$$

can be derived by starting with the trivial identity

$$a + (b+c) = a + (b+c)$$

and then using the identity

$$\square + \Delta = \Delta + \square,$$

which says that

$$b+c$$

is a name for a number, and

$$c+b$$

is *another name for that same number.* Consequently, Al writes

$$a + (b+c) = a + (\qquad).$$

Now, in the space ↗, he puts another

name for the number. Which name does he use?

14) If you have the List of identities

#1 $\square + \triangle = \triangle + \square$

#2 $\square + \triangle = \triangle + \square$

#3 $(a+b) \times (c+d) = (d+c) \times (b+a)$,

can you make up a derivation for #3, by starting with the trivial identity

$$(a+b) \times (c+d) = (a+b) \times (c+d) \,?$$

22. The Axioms

1) Suppose we spent a year writing down every identity we could think of.

Suppose we then spent a second year trying to shorten this List by every method we could think of.

What do you think the long List would look like? Try to write out some indication of what this long List would be.

2) What methods could we use to shorten this long List?

3) What do you think the final List would look like? Write some identities that you think would still be on the final List.

The following identities are known as a set of axioms:

1) $\square \times 1 = \square$

2) $\square + 0 = \square$

3) $\square \times 0 = 0$

4) $\square + \triangle = \triangle + \square$

5) $\square \times \triangle = \triangle \times \square$

6) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$

Besides this List of axioms, we have the following List of names for numbers:

$2 = 1 + 1$

$3 = 2 + 1$

$4 = 3 + 1$

$5 = 4 + 1$

$$6 = 5 + 1$$

$$\vdots$$

$$1962 = 1961 + 1$$

$$\vdots$$

Question: Why do you suppose that this List of axioms, and this List of names for numbers, are important?

23. Signed Numbers

1) Can you make up a story to correspond to $^{+}2 + {}^{+}3 = \_\_\_\_\_$?

What do you think of these stories:

2) Bill interprets $^{+}2 + {}^{+}3$ this way:

Somebody gave you \$2.

Then somebody gave you \$3.

As a result, you were \$5. richer, so:

$$^{+}2 + {}^{+}3 = {}^{+}5.$$

3) Jerry says that a certain stock went up 2 points one day. Then, the next day, it rose three points.

So for the two-day period,

the stock rose a total of 5 points.

4) Al says somebody gave you 2 pieces of candy. Then somebody gave you 3 pieces of candy. As a result you had 5 pieces of candy:

$$^{+}2 + {}^{+}3 = {}^{+}5.$$

5) Alex says:

The postman brought you a check for \$2. And then he brought you a check for \$3.

This is the same thing as if he had brought you a single check for \$5.

6) Jerry says that all good stories for $^{+}2 + {}^{+}3 = {}^{+}5$ must have this in common:

i) $^{+}2$ must be a "gain" of 2 units.

ii) ___ + ___ means that ___ + ___
↑
this

happened, and then ___ + ___
↗
this

also happened.

iii) $^{+}3$ must be a "gain" of 3 units.

Do you agree?

7) Can you make up a story for $^{+}2 + {}^{+}3$, and compare it with Jerry's description?

8) Can you make up a story for

$^{+}5 + {}^{-}2 =$ ___ ?

9) Marc suggested this story:

You found \$5.

And then you lost \$2.

Consequently, you were \$3. richer, or:

$^+5 + {}^-2 = {}^+3.$

Is this a good story?

10) Billy says that all good stories for

$^+5 + {}^-2 =$ ____

must have this in common:

i) $^+5$ must be a "gain" of 5 units.

ii) ____ + ____ means that

____ + ____

↑ this

happened, and then

____ + ____

this ↑

happened.

iii) $^{-}2$ must be a "loss" of 2 units.

11) Can you make up a story for $^{+}5 + ^{-}2 = \_\_\_$, and compare it with Billy's description?

Can you make up stories for each of these?

12) $^{+}8 + ^{+}11 = \_\_\_$

13) $^{+}3 + ^{+}7 = \_\_\_$

14) $^{-}2 + ^{-}5 = \_\_\_$

15) $^{+}9 + ^{-}2 = \_\_\_$

16) $^{-}3 + ^{+}7 = \_\_\_$

17) $^{+}5 + ^{-}8 = \_\_\_$

18) $^{-}2 + ^{-}10 = \_\_\_$

19) When we come to subtraction problems,

instead of having

$$\underset{\underset{\text{this}}{\uparrow}}{\underline{\qquad\qquad}} + \underline{\qquad\qquad}$$

happen, and then

$$\underline{\qquad\qquad} + \underset{\underset{\text{this}}{\uparrow}}{\underline{\qquad\qquad}}$$

happen, we need a different interpretation.

We need to have

$$\underset{\underset{\text{this}}{\uparrow}}{\underline{\qquad\qquad}} + \underline{\qquad\qquad}$$

happen, and then we need to "undo"

$$\underline{\qquad\qquad} - \underset{\underset{\text{this,}}{\uparrow}}{\underline{\qquad\qquad}}$$

or have it "unhappen."

Can you make up a story for

$$^+5 - {}^+2 = \_\_\_\_ ?$$

20) Jerry made up a story: last week you found \$2 (a "gain" of 2 units, or $^+2$). That has already been recorded in your balance.

Today, however, you received a gift of \$5, and you also found the rightful owner of the \$2, so you gave it back:

$^+5 - \_\_\_\_$

↑

a gain of \$5.

$\_\_\_ - \_\_\_\_$

↑

but at the same time, "undoing"

$\_\_\_ - \underline{{}^+2}$

↑

an earlier gain of \$2.

So that the result of the two transactions today was a gain of $3:

$$^{+}5 - ^{+}2 = ^{+}3$$

Do you agree with Jerry's interpretation?

21) Mary says there were actually three transactions in Jerry's story. What were they?

22) Which one did not count in the calculation? Why not?

23) Henry says Jerry's answer should have been $^{+}5$, because the $2 was found, then returned, so it cancelled out. Do you agree?

24) Jerry says if you're going to "undo" some earlier transaction, you need to explain what it was, but since it has already been recorded, all you

record *now* is the operation of *undoing* it.

Do you agree?

25) Marie says that the *three* transactions would add like this:

$${}^+2 + {}^+5 - {}^+2 = {}^+5,$$

but that the *finding of the $2* happened last week

$${}^+2 + {}^+5 - {}^+2 = \underline{\qquad}$$

(recorded last week)

and was recorded then.

What happened *today* was:

$${}^+5 - \underbrace{{}^+2} = {}^+3$$

receiving gift of $5

returning the $2 (from last week) to its rightful owner

So, the result of today's transaction should be $^{+}3$.

Do you agree?

26) Bart made up this story:

"Five years ago a thief robbed me of $500 in rare old coins and escaped before I could catch him.

"Today I saw the thief on Main Street. I called a policeman, and with his help we captured the thief.

"I received a $100 reward, and I also recovered my missing $500 in old coins.

"I can write today's results like this:

$^{+}100$

a gain of $100

$^{+}100 - {}^{-}500$

plus "undoing" a previous loss of $500

"So the results of today's activity is

$$^{+}100 - {}^{-}500 = {}^{+}600,$$

a gain of $600."

Is Bart's story a correct interpretation?

27) Five years ago you found $3.

Today you earned $25, but you also found the owner of the $3, and returned it.

How many transactions were there today?

Can you write an equation for today's transactions?

28) Three years ago you lost $5.

Today you earned $10, and you also had the missing $5 returned to you.

How many transactions are described here?
How many transactions were there today?
Can you write an equation for today's transactions?

29) How does the story for

$^{+}5 + {}^{-}2 =$ ____

differ from the story for

$^{+}5 + {}^{+}2 =$ ____ ?

30) How does the story for

$^{+}8 - {}^{-}2 =$ ____

differ from the story for

$^{+}8 + {}^{+}2 =$ ____ ?

Are the final answers the same?

31) Diana says that every good story for

subtraction

____ − ____

must have ____ − ____

this↗

"unhappening" or being undone somehow.

How many ways can you think of to explain something being undone?

32) Marie says if money was lost, this past loss can be undone if today the money was returned. Can you make up a story like this to explain $^{+}7 - {}^{-}2 =$ ____?

What happened today?

What had happened at some time in the past?

What was the result of today's transactions?

33) Al says that if you were robbed last month, this loss (represented by a negative number) can be undone if, today, the police recover the stolen money and return it to you.

Can you use this kind of story to explain

$^{+}5 - {}^{-}3 =$ ____ ?

34) Bill says that \$5 received in the past is a past gain. It could be *undone*, *today*, if it turned out that you had been given a bad check.

Can you use this idea to make up a story for

$^{+}8 - {}^{+}3 =$ ____ ?

What happened *today*?

What had happened previously?

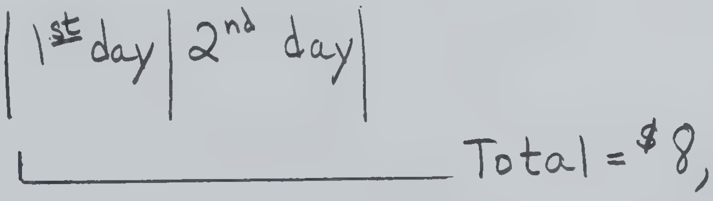
Does $^{+}8 - {}^{+}3 =$ ____ represent *only* *today's transactions*, or does it represent *all* *three* transactions?

35) Linda says you can represent $^{+}2$ as a check for \$2, $^{-}5$ as a bill for \$5, *addition* as

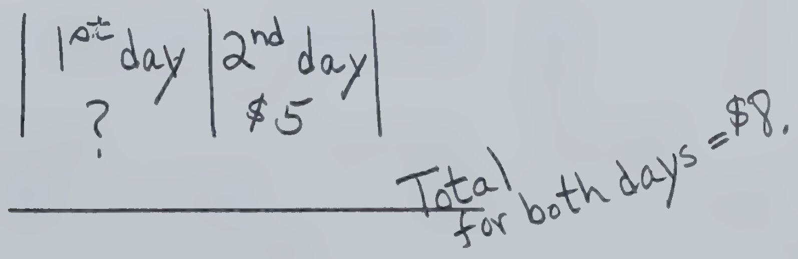
mail which the postman brings, and subtraction as mail which the postman takes away (because yesterday he delivered it to you by mistake).

Can you make up some addition and subtraction problems, and explain them by "postman" stories?

36) Jerry says there is another way to "undo" a past gain or loss. Suppose, for example, that you made \$8 in two days



and you know that you made \$5 yesterday



In order to find how much you earned two days ago, you look at the total earned ($8), and "undo" what you earned yesterday ($5). The result must be what you earned two days ago.

Can you write an equation for this story?

37) Can you make up a story of this same kind to explain

$$^{-}7 - {}^{-}2 = \underline{\qquad}\,?$$

38) Can you make up a story like Jerry's in problem # 36 to explain this:

$$^{+}8 - {}^{-}3 = \underline{\qquad}\,?$$

39) What does Marie mean by saying that, to subtract, you "just change the sign and add"?

Does Marie's rule really work?

24. Using Identities

1) Is this an identity?

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

2) Jerry says we can multiply

$$4 \times 2\tfrac{1}{4}$$

as follows:

$$2\tfrac{1}{4} = \frac{9}{4}$$

$$4 \times \frac{9}{4} = \frac{36}{4} = 9$$

Is Jerry's method correct?

3) Alan says we can multiply $4 \times 2\tfrac{1}{4}$ as follows:

$$4 \times 2 = 8$$

$$4 \times \frac{1}{4} = 1$$

$$\begin{array}{r} 8 \\ +\,1 \\ \hline 9 \end{array}$$

Is Alan's method correct?

4) Elizabeth says Jerry used one side of the "distributive law" identity, and Alan used the other side.

Which side did Jerry use?

Which side did Alan use?

Is Elizabeth right?

Multiply, by two different methods:

5) $8 \times 7\frac{1}{8} =$ ___

6) $6 \times 2\frac{1}{3} =$ ___

7) $16 \times 2\frac{1}{2} =$ ___

8) $9 \times 5\frac{2}{9} =$ ___

9) $7 \times 2\frac{3}{7} =$ ___

10) Is this an identity? $\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$

25. Open Sentences and Signed Numbers

Can you multiply these signed numbers?

1) ${}^{+}2 \times {}^{-}3 = \_\_\_\_$

2) ${}^{-}2 \times {}^{-}5 = \_\_\_\_$

How good are you at discovering things in science and mathematics? Can you find the truth sets for these?

3) $(\square \times \square) - (13 \times \square) + 22 = 0$ $\{\ ,\ \}$

4) $(\square \times \square) - (5 \times \square) + 6 = 0$ $\{\ ,\ \}$

5) $(\square \times \square) - (9 \times \square) + 14 = 0$ $\{\ ,\ \}$

6) $(\square \times \square) - ({}^{+}2 \times \square) + {}^{-}15 = 0$ $\{\ ,\ \}$

7) $(\square \times \square) - ({}^{+}3 \times \square) + {}^{-}10 = 0$ $\{\ ,\ \}$

8) $(\square \times \square) - ({}^{+}5 \times \square) + {}^{-}14 = 0$

$\{\ ,\ \}$

9) $(\square \times \square) - (15 \times \square) + 50 = 0$

$\{\ ,\ \}$

10) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{+}6 = 0$

$\{\ ,\ \}$

26. Graphs

Have you discovered the secret pattern for making these graphs quickly? You can complete and plot one point, and get all the other points by using the secret pattern.

Don't tell anybody else about the secret pattern!

1) For the open sentence

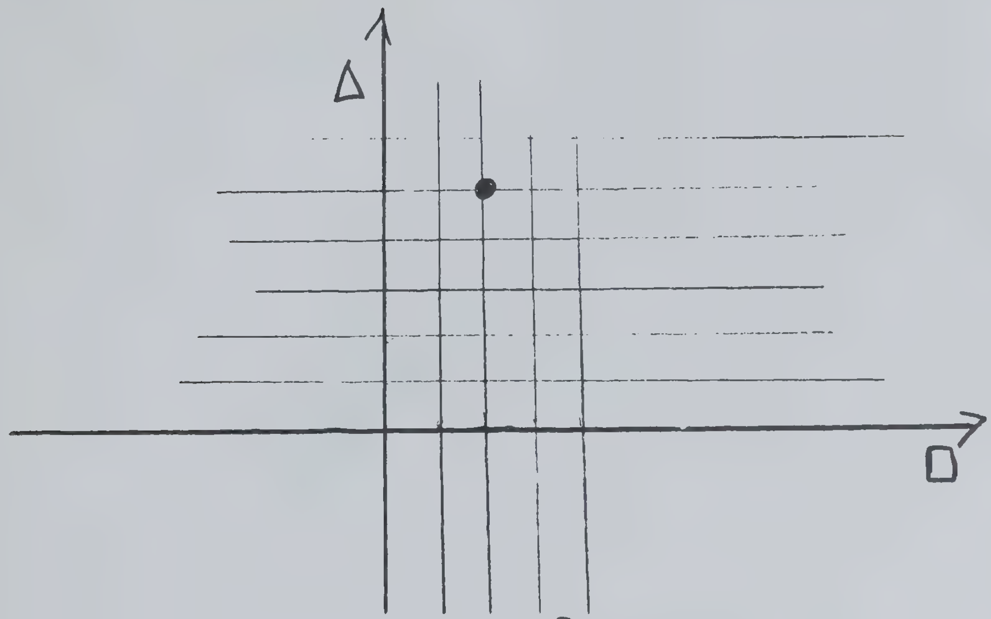
$$\Delta = (2 \times \Delta) + 1,$$

David completed this table:

□	Δ
2	5

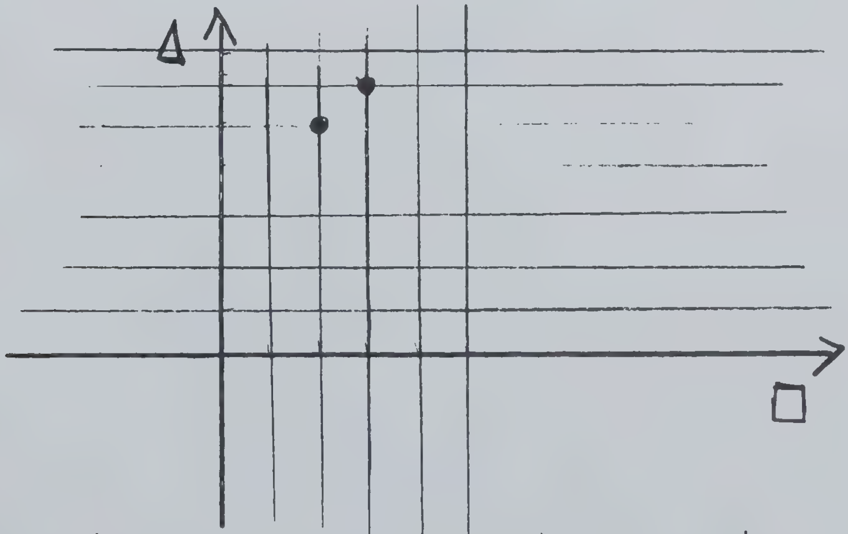
Is this correct?

2) Jerry plotted (2,5) like this:



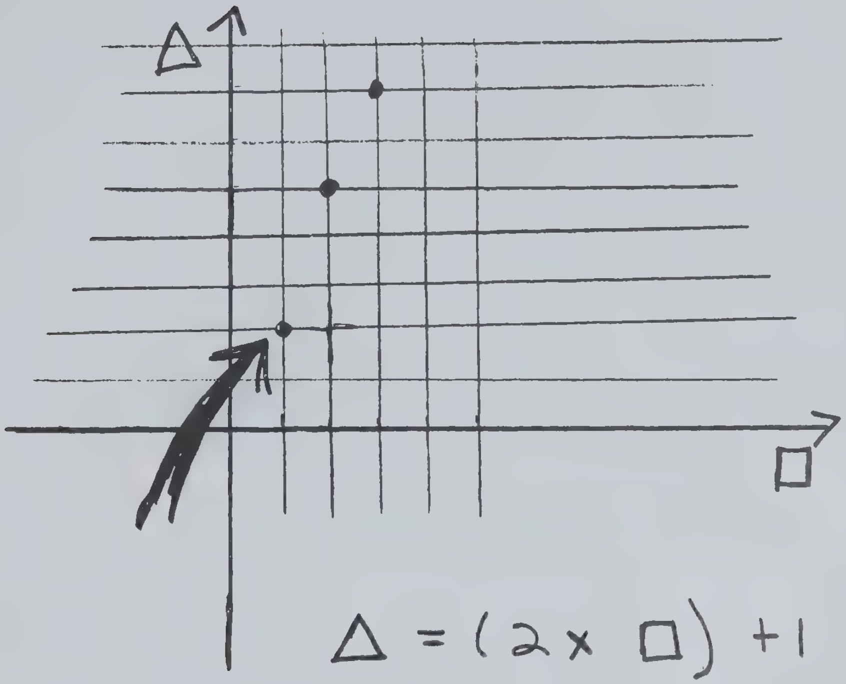
Is this correct?

3) Marie says she knows the secret pattern. She says another point should be here:



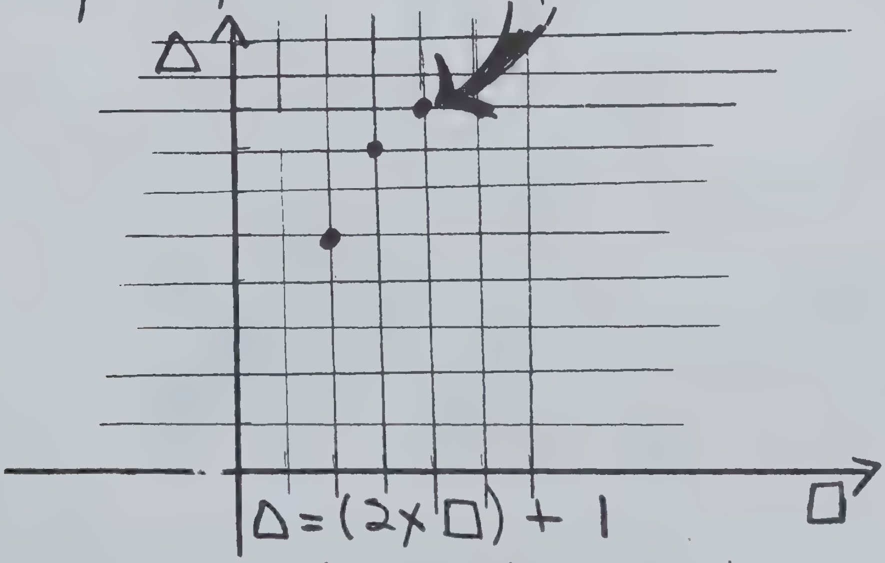
Does Marie really know the secret?

4) Al says another point should go here:



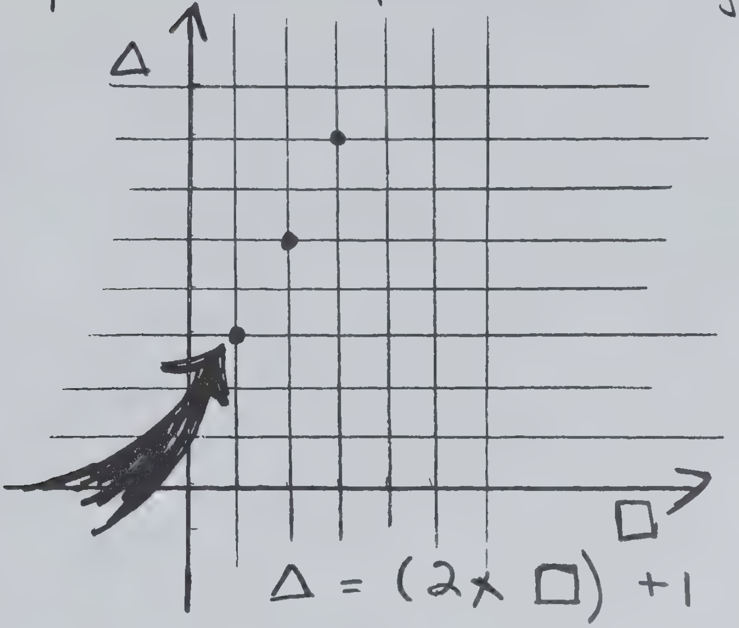
Does Al know the secret pattern?

5) Nancy says another point should go here:



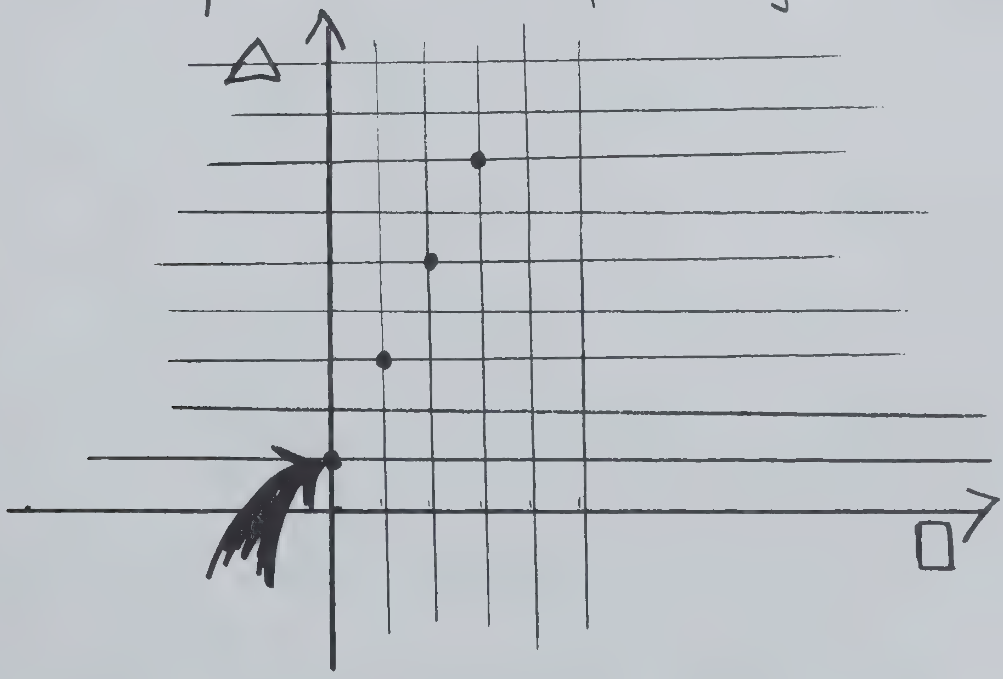
Do you think Nancy knows the secret?

6) Jeff says another point should go here:



Does Jeff know the secret?

7) Bill says another point goes here:



Does Bill know the secret pattern?

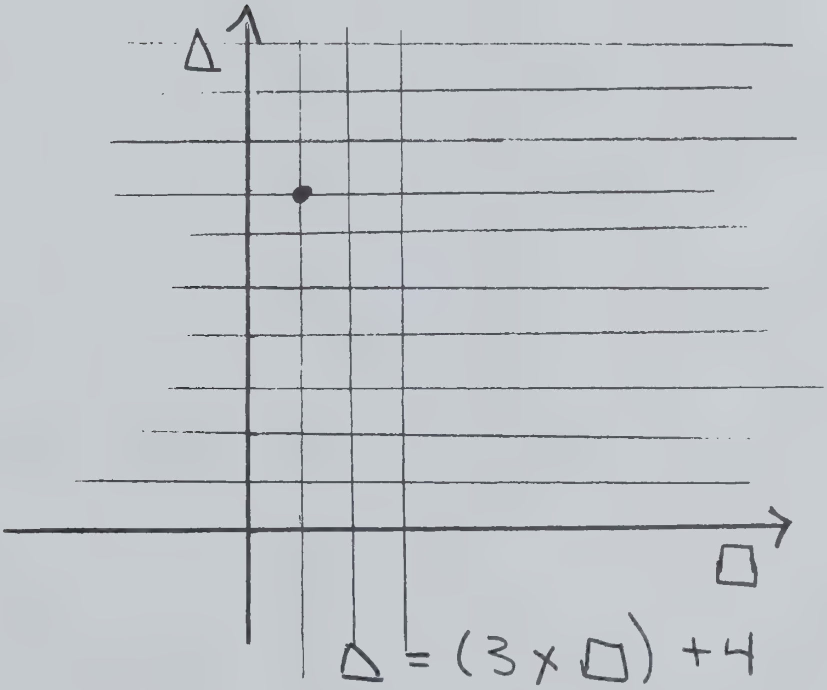
8) For the open sentence

$$\triangle = (3 \times \square) + 4,$$

is this table correct:

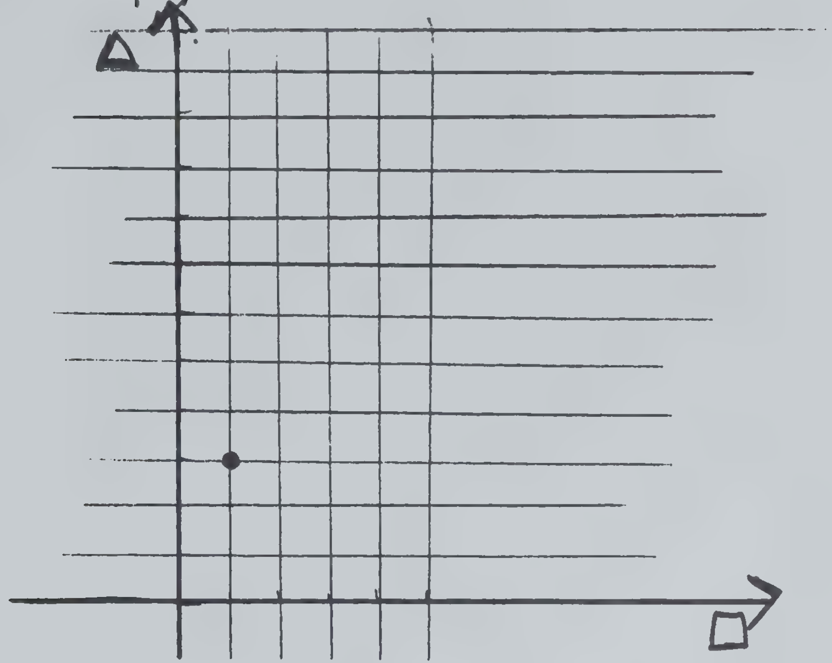
$\square$	$\triangle$
1	7

9) Is this point correct?



10) Can you use the secret pattern to find five more points on this graph, without computing them?

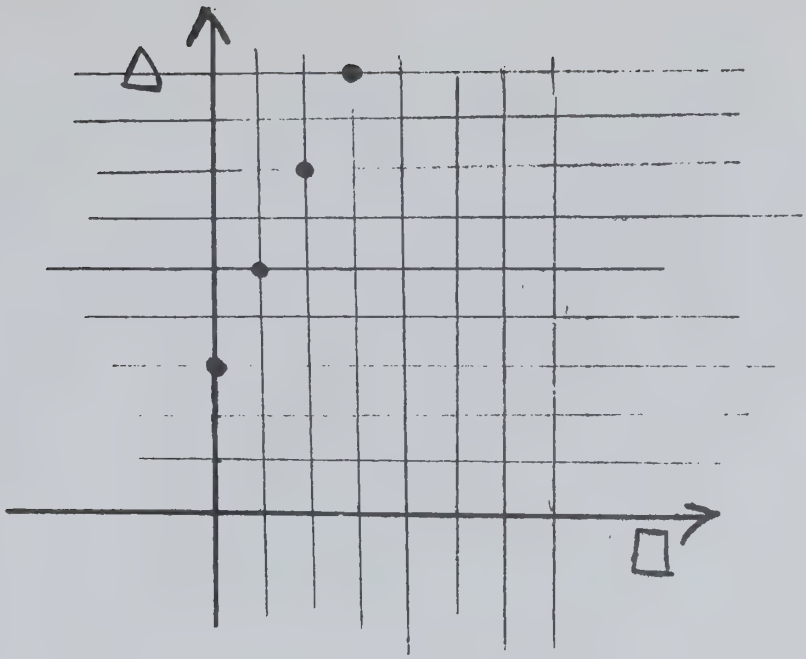
11) Is this point correct?



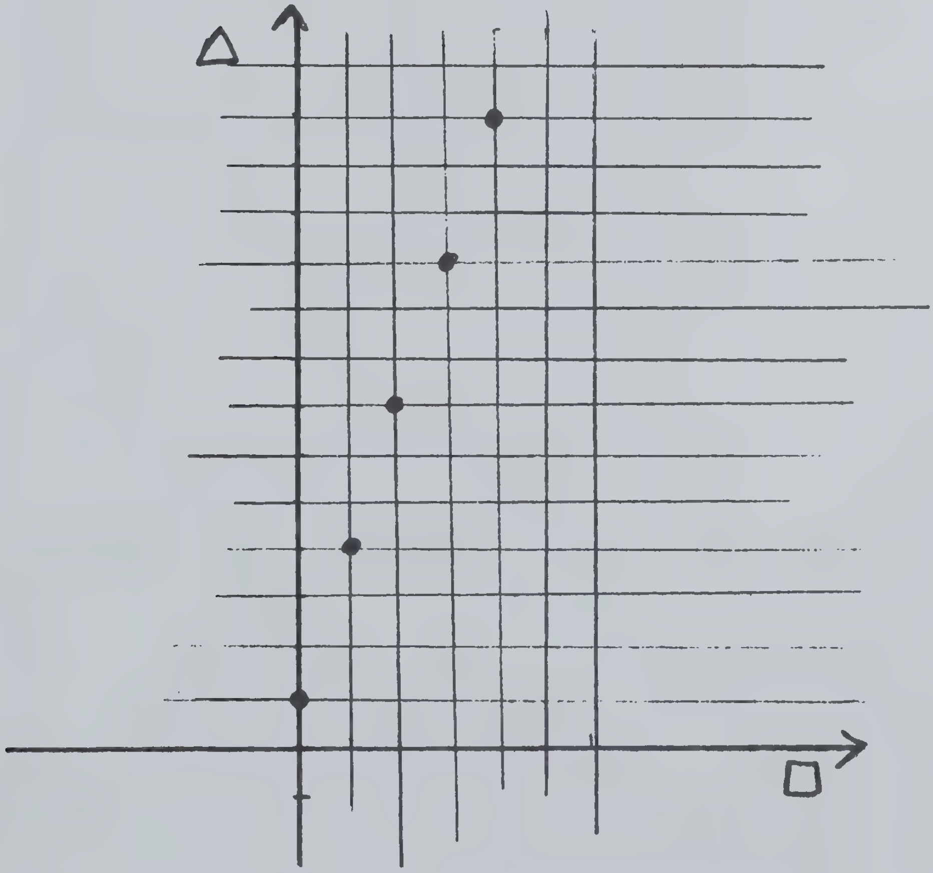
$\triangle = (5 \times \square) - 2$

12) Can you use the secret pattern to find five more points on this graph?

13) Can you write the open sentence whose truth set is indicated here:

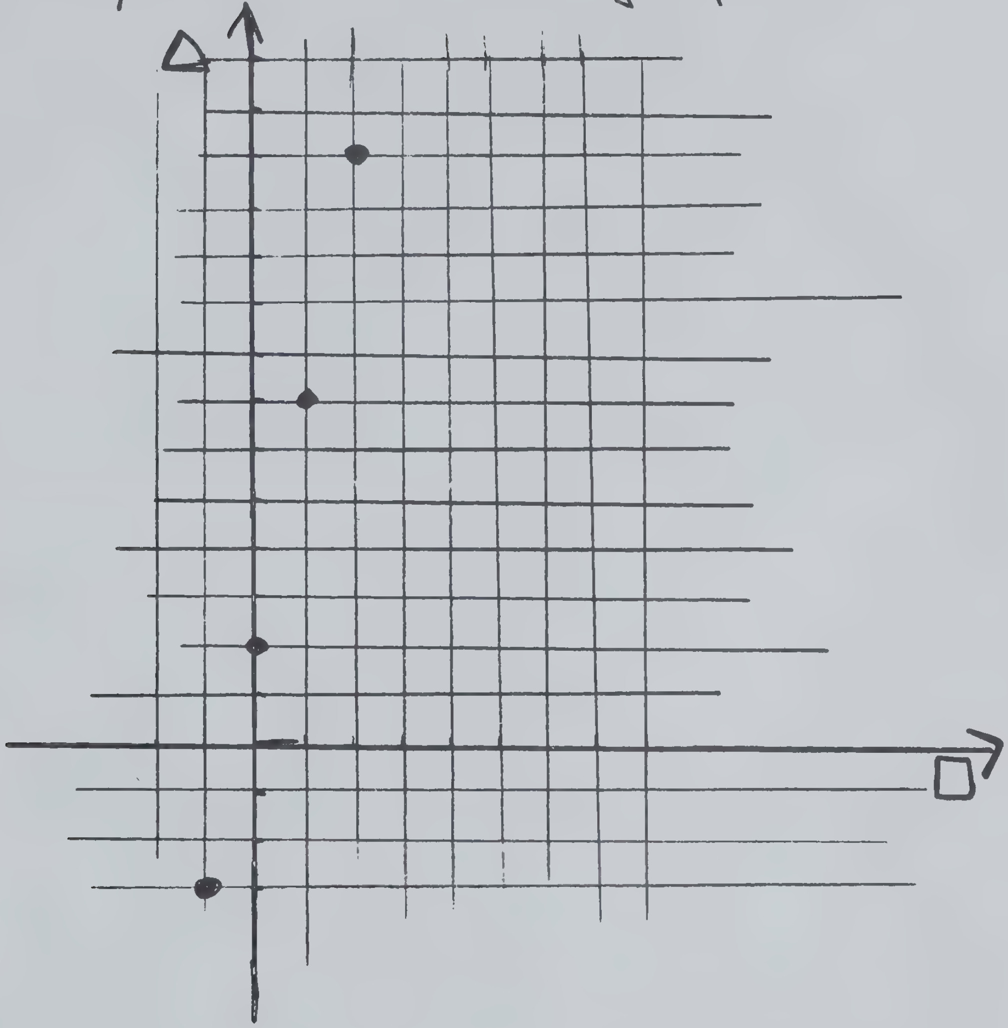


14) Dave made this graph:



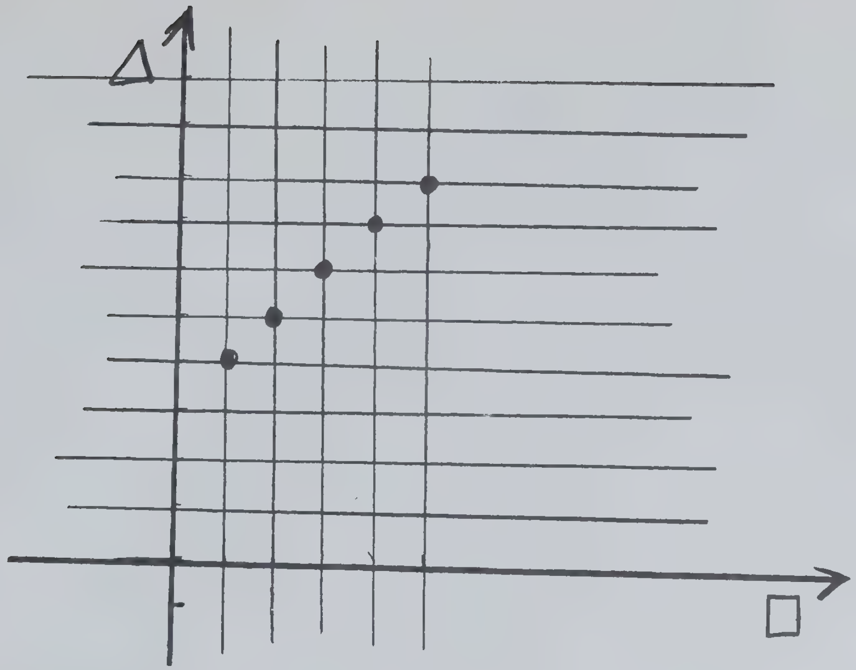
What equation was Dave using?

15) Jerry made this graph:



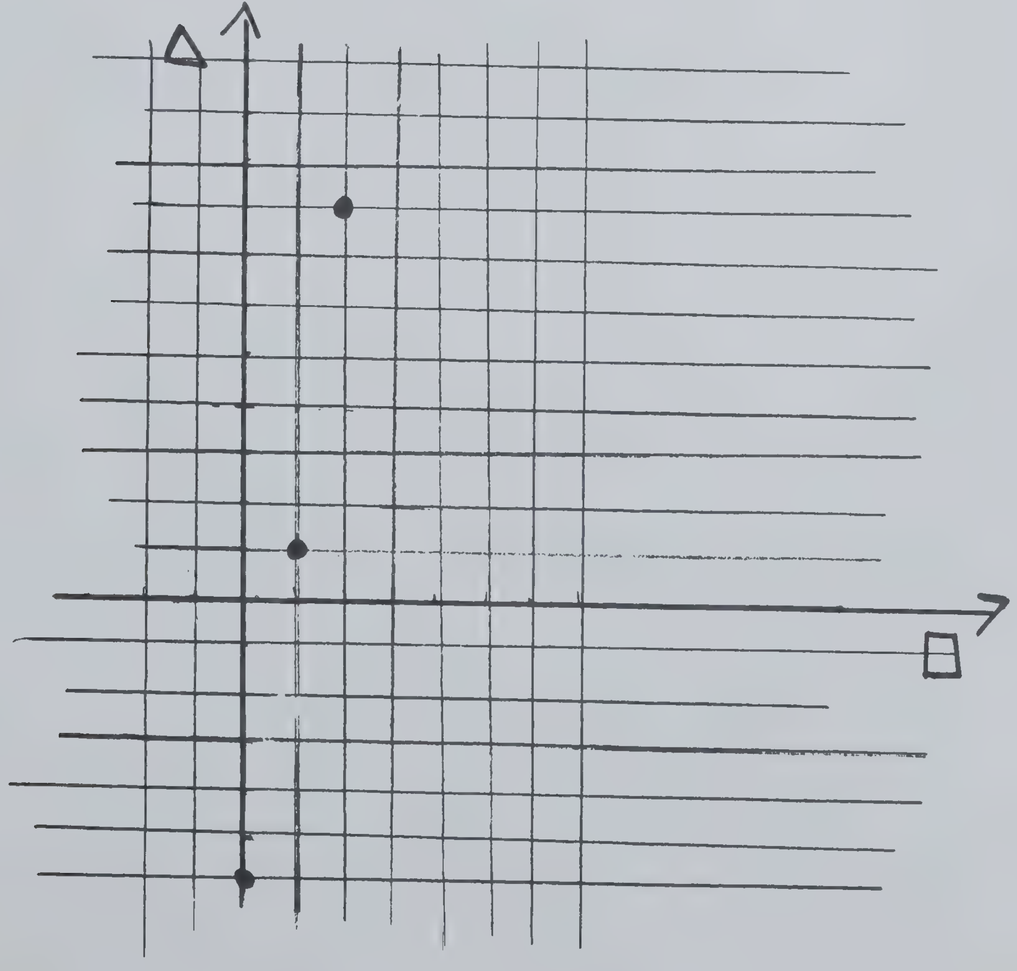
What equation was Jerry using?

16) Bill made this graph:



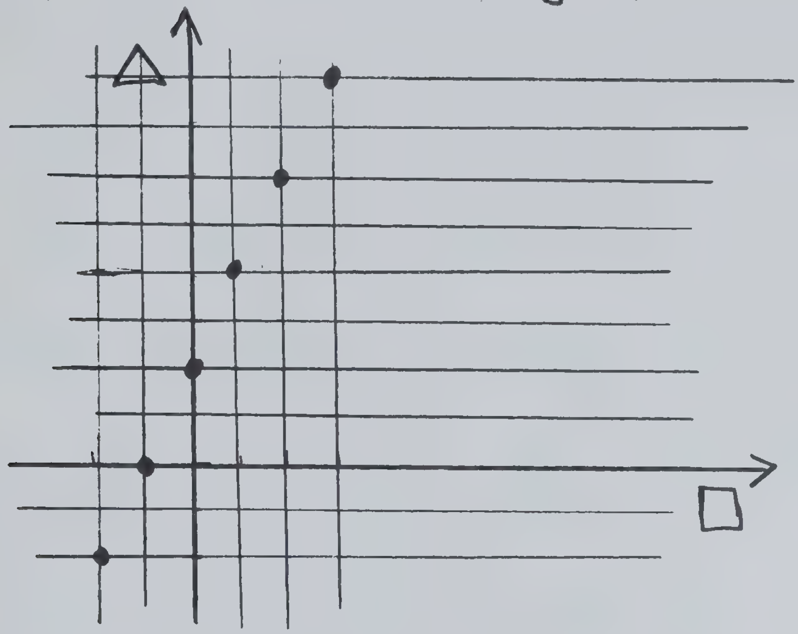
What equation was Bill using?

17) Joan made this graph:



What equation was Joan using?

18) Ellen made this graph:



What equation was Ellen using?

27. Equations

1. Can you find the truth set for this open sentence?

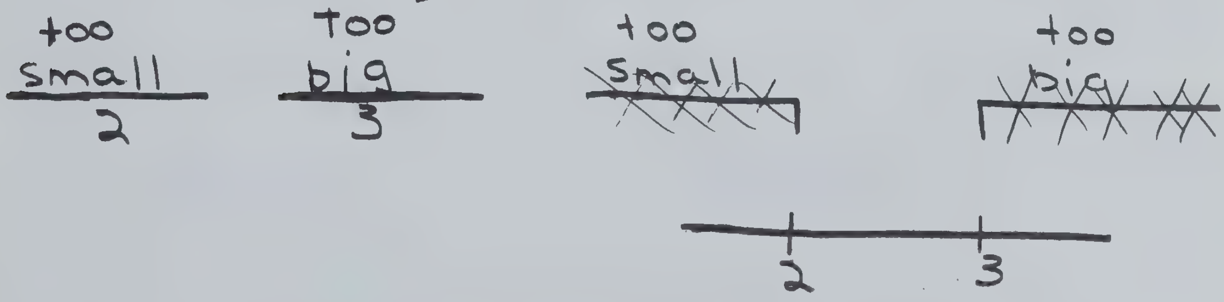
$$3 + \square = 5$$

2. Without finding the answer for this equation, can you guess whether it will be a whole number or a fraction?

$$3 + (11 \times \square) = 28$$

3. Can you prove it?

4. Bill uses this picture to help him solve the equation $3 + (11 \times \square) = 28$:



Does this picture help you?

5. Al says that he thinks $2\frac{1}{2}$ may be the answer for this equation.

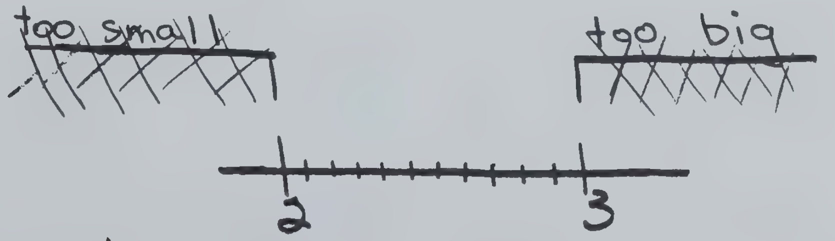
Without computing, can you explain why $2\frac{1}{2}$ cannot be the answer to this equation?

6. In the equation $3+(11 \times \square) = 28$, is $(11 \times \square)$ a fraction or a whole number?

7. What denominator must the answer have?

8. Marie says that the answer must be a fraction, between 2 and 3, with a denominator of 11. Do you agree with Marie?

9. Bill's picture now looks like this:



Does this picture help you guess the answer?

10. Jerry tried $2\frac{9}{11}$ in the equation. Was it too small or too big?

11. After Jerry had tried $2\frac{9}{11}$, how should we draw Bill's picture?

12. Can you find a number that is too small? How does this change Bill's picture?

13. Can you find a number that is too big? How will you change Bill's picture now?

14. Can you find the truth set for the equation $3 + (11 \times \square) = 28$?

Can you find the truth sets for these open sentences:

15. $5 + (13 \times \square) = 29$

16. $3 + (7 \times \square) = 12$

17. $9 + (13 \times \square) = 50$

18. $3 + (17 \times \square) = 88$

Two open sentences are called equivalent if they have the same truth set.

19. Is the open sentence $2 + \square = 5$ equivalent to the open sentence $2 \times \square = 6$?

20. Are these equivalent equations:

$$5 + (2 \times \square) = 13$$

$$6 + (2 \times \square) = 14$$

21. Can you make up an equation that is equivalent to this one:

$$3 + (2 \times \square) = 15$$

22. Can you make up a pair of equivalent equations?

If an operation changes an equation, but does not change the truth set, we will call it a transformation operation.

23. Henry started with this equation:

$$3 + \square = 5 .$$

He found an equivalent equation that began $4 + \_\_\_\_\_\_\_\_$.

Can you guess the equation that Henry found?

24. Can you use Henry's method to start with $3 + \square = 5$ and get an equation that begins $7 + \_\_\_\_\_\_\_\_$?

25. Can you use Henry's method to get 5 more equations, each of which is equivalent to $3 + \square = 5$?

26. Can you describe Henry's method?

27 Can you invent some method of your own that is different from Henry's method?

28. Laura invented this method:

She started with this equation:

$$7 + \square = 23$$

and found this equivalent equation:

$$\square + 7 = 23.$$

Is Laura's method correct?

29. Can you use Laura's method to get a pair of equivalent equations?

30. Paul says he used Laura's method this way: He started with:

$$3 + \square = 25;$$

then he used Laura's method to get

$$\square \times 3 = 25.$$

Are these two equations equivalent?

31. Can you use Laura's method to get a pair of equivalent equations?

32. Can you describe Laura's method?

33. Can you describe Henry's method?

34. Can you find an equation that is equivalent to this one:

$$3 + (11 \times \square) = 61$$

35. Alan has two methods: he says that "Alan's First Method" works like this: You can start with this equation:

$$8 + (3 \times \square) = 33$$

and you can use "Alan's First Method" to get

$$16 + (6 \times \square) = 66.$$

Does "Alan's First Method" really work?

36. Can you start with this equation:

$$3 + (8 \times \square) = 21$$

and get an equivalent equation that begins this way: $4 + $ ________ ?

Which method did you use?

37. Can you start with $3 + (8 \times \square) = 21$, and get an equation that begins

$$(8 \times \square) + 3 = \underline{\qquad\qquad} ?$$

Which method did you use?

38. How many kinds of transformation operations do you know?

39. What do we mean by equivalent equations?

40. What do we mean by a transforming operation?

41. Can you guess what "Alan's Second Method" is?

42. Alan says his "Second Method" is the reverse of his "First Method". Can you guess what his "Second Method" is?

43. Can you start with the equation

$$8 + (16 \times \square) = 42$$

and get the equivalent equation

$$4 + (8 \times \square) = 21 ?$$

Which method did you use?

44. Let us start with this equation and see how many equivalent equations we can get:

$$3 + (11 \times \square) = 50$$

45. Lori got this:

$$4 + (11 \times \square) = 51.$$

What method did she use?

46. Ginger went on to get this:

$$5 + (11 \times \square) = 52.$$

what method did Ginger use?

47. Then Joan got this:

$$6 + (11 \times \square) = 53.$$

What method did Joan use?

48. Mary got this:

$$4 + (12 \times \square) = 54.$$

What method did Mary use?

49. Jerry got this:

$$12 + (22 \times \square) = 106.$$

What method did Jerry use?

50. Al got this:

$$11 + (22 \times \square) = 105 .$$

What method did Al use?

51. Wayne likes "fancy methods". He changed Al's equation into this one:

$$1 + (2 \times \square) = 9\tfrac{6}{11} .$$

Can you see how Wayne did it?

52. Tony got this from Wayne's equation:

$$0 + (2 \times \square) = 8\tfrac{6}{11} .$$

How did he do it? Did he really do it correctly?

53. Mary says we can rewrite Tony's equation this way:

$$2 \times \square = 8\tfrac{6}{11} .$$

Do you agree?

54. Jerry says he can use transforming operations to help him find the truth set for the equation

$$5 + (13 \times \square) = 36.$$

Can you guess how Jerry does it?

55. Marie says we can change Jerry's equation into an easier one that begins

4 + ________ .

Do you agree?

56. Can you change Marie's result into a slightly simpler equation that begins

3 + ________ ?

57. Can you make this equation still simpler?

58. Can you find the truth set?

28. Think of a Number

1) What is $^{-}2 \times {}^{+}3 =$ ____?

2) What is $^{-}2 \times {}^{-}5 =$ ____?

3) The truth set for this equation contains two numbers. Can you find them?

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

Can you find the truth sets for these equations?

4) $(\square \times \square) - (13 \times \square) + 22 = 0$

5) $(\square \times \square) - (20 \times \square) + 96 = 0$

6) $(\square \times \square) - (20 \times \square) + {}^{-}15 = 0$

7) $(\square \times \square) - ({}^{-}1 \times \square) + {}^{-}6 = 0$

8. Gene: Think of a number. #

Beryl: All right.

Gene: Multiply it by itself. # x #

Beryl: Uh-huh ...

Gene: Now subtract twice the number you started with. +2 x #

Beryl: Yes ...

Gene: Now add 11. +11

Beryl: Uh-huh ...

Gene: ... and tell me the answer.

Beryl: Fourteen...

Gene: The number you thought of was 3.

Beryl: Yes, that was it.

9. Beth: Jerry, think of a number. $\#$

Jerry: OK....

Beth: Multiply it by itself $\# \times \#$

Jerry: Uuh-huh...

Beth: Subtract twice the number you started with $-2 \times \#$

Jerry: Uuh-huh ...

Beth: ...and add 5. $+5$

Jerry: Uuh-huh...

Beth: ...and tell me the result.

Jerry: Eighty-five

Beth: Was 10 the

number you started with?

Jerry: Yes, it was.

10. Marie says that mathematics tells you that Jerry started with either $^{+}10$ or else with $^{-}8$. In order to guess whether Jerry started with $^{+}10$ or -8, you have to use psychology.

What does Marie mean?

11. While Beth and Jerry were talking, Ann-Marie wrote this:

$$(\square \times \square) - (2 \times \square) + 5 = 85.$$

Does this help you to find the number that Jerry started with?

12. Al says that he wants a zero on the right hand side of the equation.

How can he get it there?

13. Lori says that Al can subtract .85 from each side of the equation, getting

$$(\square \times \square) - (2 \times \square) + {}^{-}80 = 0.$$

Do you agree with Lori?

14. Tony says the truth set for the equation

$$(\square \times \square) - (2 \times \square) + {}^{-}80 = 0$$

is $\{{}^{+}10, {}^{-}8\}$. Is Tony right?

15. Can you make up a problem like this?

29. Graphs

1) Can you make a table to show the truth set of

$$\triangle = (3 + \square) + 2 \; ?$$

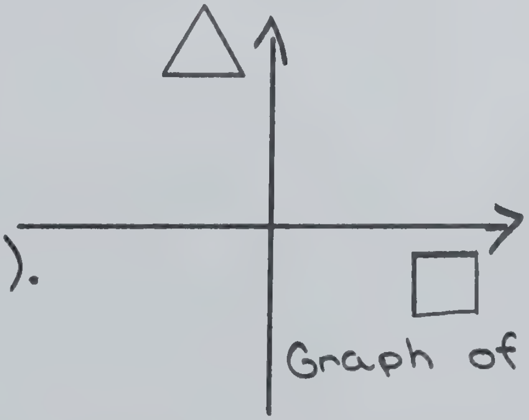
$\square$	$\triangle$

Table for truth set

Use only whole numbers (Either positive or negative).

2) Can you make a graph to show the truth set for $\triangle = (3 \times \square) + 2$?

Use only whole numbers (either positive or negative).



Graph of Truth Set

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3) What is the "pattern" of this graph?

4) Without doing any arithmetic, can you make a new point on your truth set graph?

5) Without doing any arithmetic, can you make a point on this truth set graph that corresponds to $\boxed{\frac{1}{2}}$, $\overset{?}{\triangle}$.

(Locate the point without doing arithmetic by looking at the geometry of the graph!)

6) Now do the arithmetic, and see if your point was in the right place. Was it?

7) In the same way, look at the geometry of the graph and mark a point of the truth set corresponding to

$\boxed{1\frac{1}{2}}$, $\overset{?}{\triangle}$.

8) Now do the arithmetic:

$$\triangle = \left(3 \times \boxed{1\frac{1}{2}}\right) + 2$$

Was your point in the right place?

9) By looking at the geometry of the graph, mark a point of the truth set corresponding to

$\boxed{2\frac{1}{2}}$, $\overset{?}{\triangle}$.

10) By using the geometry (instead of arithmetic), mark these points on the truth set graph:

a) $\boxed{3\frac{1}{2}}$, $\overset{?}{\triangle}$

b) $\boxed{-\frac{1}{2}}$, $\overset{?}{\triangle}$

c) $\boxed{2\frac{1}{4}}$, $\overset{?}{\triangle}$

d) $\boxed{2\frac{3}{4}}$, $\overset{?}{\triangle}$

e) $\boxed{1\frac{1}{4}}$, $\overset{?}{\triangle}$

f) $\boxed{1\frac{3}{4}}$, $\overset{?}{\triangle}$

g) $\boxed{1\frac{1}{8}}$, $\overset{?}{\triangle}$

h) $\boxed{2\frac{1}{3}}$, $\overset{?}{\triangle}$

i) $\boxed{3\frac{1}{10}}$, $\overset{?}{\triangle}$

11) Suppose you marked all of these, and other similar points of the truth set.

Can you describe what the truth set graph would look like?

12) Using only whole numbers (both positive and negative if you wish), make a graph for the truth set of

$$\triangle = (4 \times \square) + 1$$

13) What is the "pattern" of this graph?

14) Jerry says the "pattern" of the graph for $\triangle = (4 \times \square) + 1$

is "over one (to the right), and up three".

Do you agree?

15) Look at the graph for $\triangle = (4 \times \square) + 1$. Without doing any arithmetic, can you fill in some points corresponding to fractions?

16) If all of these points were filled in, can you describe what the graph would be?

17) If your graph is very carefully drawn, take a ruler and complete it. Can you use your graph to find the number that should go in the triangle in order to make a true statement: $\triangle = (4 \times \boxed{1\frac{1}{3}}) + 1$?

18) Can you use your graph to find the number that should go in the triangle:

$$\triangle = \left(4 \times \boxed{2\tfrac{1}{3}}\right) + 1 \text{ ?}$$

30. A Discovery by Peter

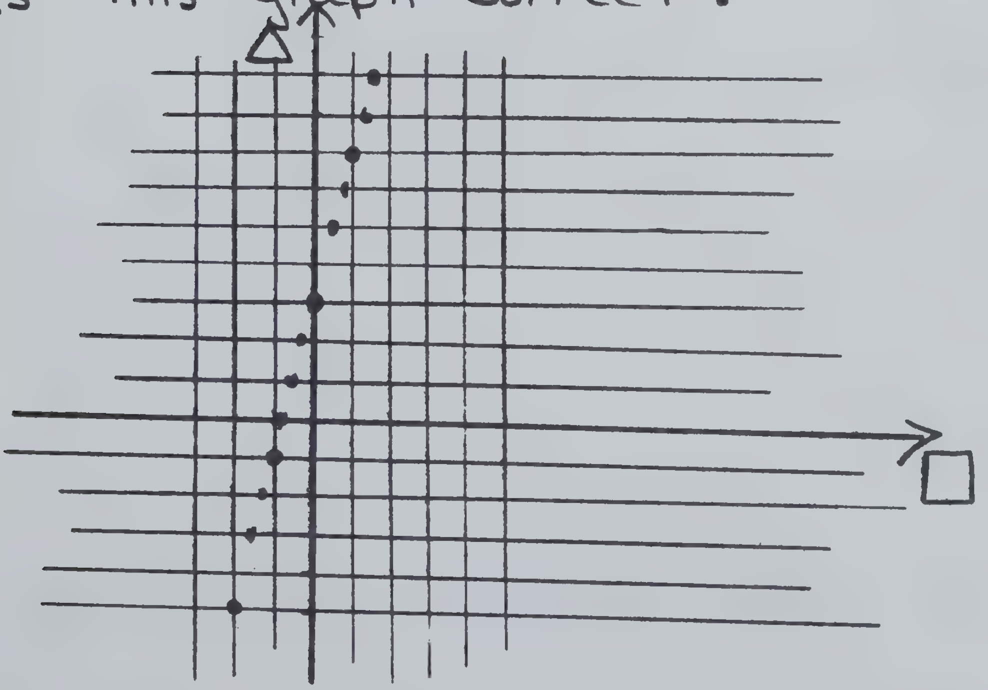
1) For the open sentence

$$\triangle = (4 \times \square) + 3,$$

is this table for the truth set correct?

$\square$	$\triangle$
0	3
1	7
2	11
3	15
$^{-}1$	$^{-}1$
$^{-}2$	$^{-}5$

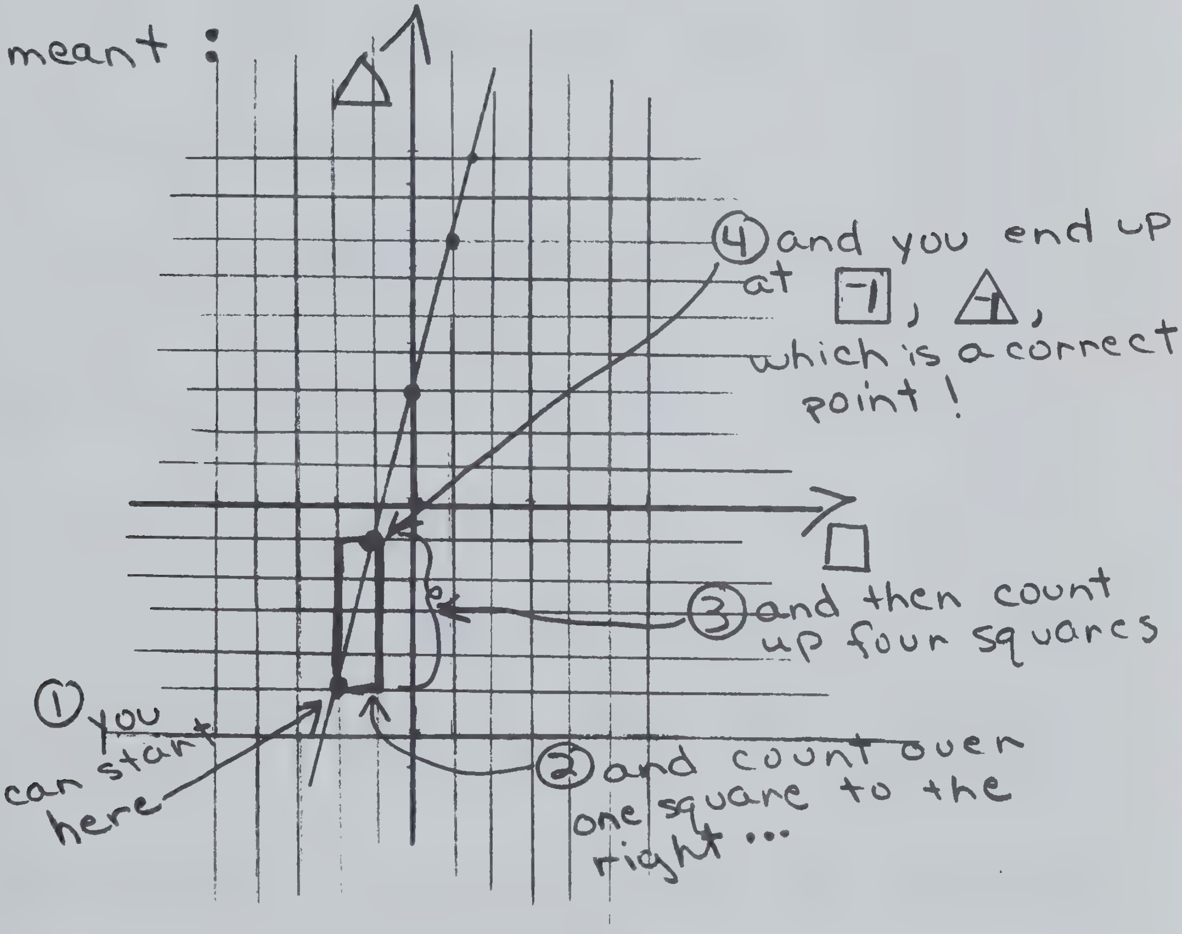
2) Is this graph correct?



3) Debbie says that there is a pattern of "over one square to the right, up four squares."

Do you agree?

4) Debbie made this picture to show what she meant:


△
□
④ and you end up at □-1, △-1, which is a correct point!
③ and then count up four squares
① you can start here
② and count over one square to the right ...

Does Debbie's "pattern" really work?

5) Would Debbie's pattern work if you started at $\boxed{0}$, $\triangle_{3}$?

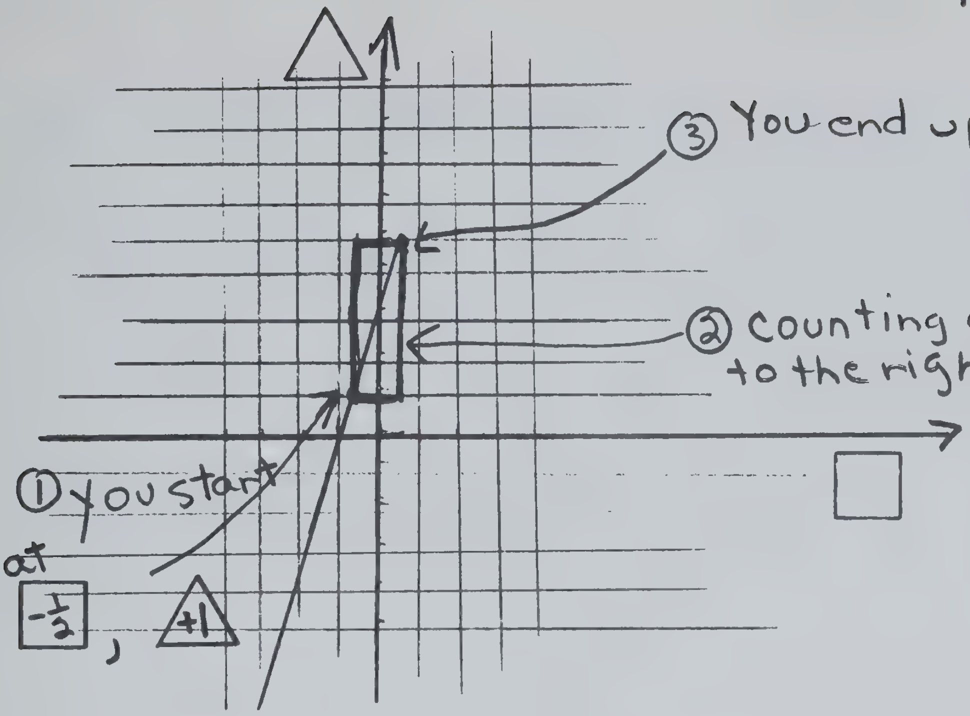
6) Would Debbie's pattern work if you started at $\boxed{-1}$, $\triangle_{-1}$?

7) Does $\boxed{-\frac{1}{2}}$, $\triangle_{+1}$ belong to the truth set for the open sentence

$$\triangle = (4 \times \square) + 3 \; ?$$

8) Does Debbie's pattern work if you start at $\boxed{-\frac{1}{2}}$, $\triangle_{+1}$?

9) Bill says that Debbie's pattern does work when you start at $\boxed{-\frac{1}{2}}$, $\triangle_{+1}$. Bill drew this diagram to show how he counted "over one square to the right, the up four squares":



③ You end up at $\boxed{+\frac{1}{2}}$, $\triangle 5$

② counting over one square to the right, and up four

① you start at $\boxed{-\frac{1}{2}}$, $\triangle +1$

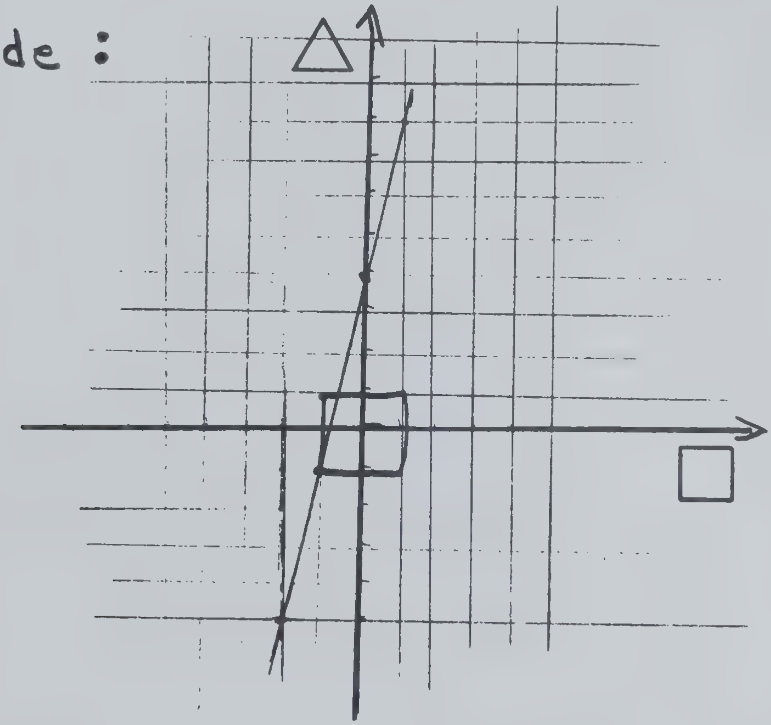
Bill says that $\boxed{+\frac{1}{2}}$, $\triangle 5$ does belong to the truth set for the open sentence.

Do you agree with Bill?

10) Can you prove that $\boxed{+\frac{1}{2}}$, $\triangle 5$ belongs to the truth set for the open sentence?

11) Peter said: "I've been thinking. Debbie's pattern works no matter what size of square you use." Is Peter right?

12) Peter made several pictures to show what he meant. First, he started at $\boxed{-1}$, $\triangle{-1}$, and used a square that was 2 units on each side:

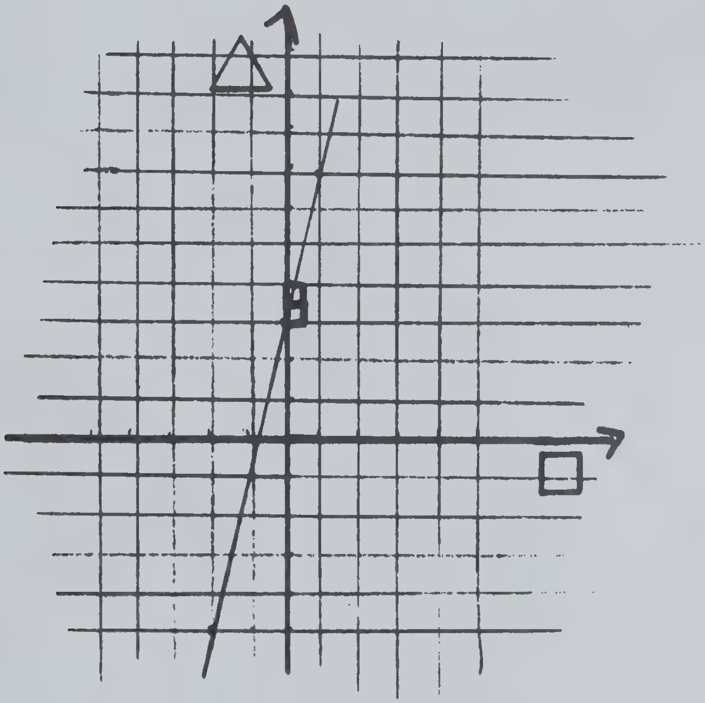


This is one of the squares that Peter used. Can you finish this diagram?

13) When you counted "over one, up four," using Peter's big squares, where did you end? Was this a point of the truth set?

14) Can you describe Peter's discovery?

15) Peter also made a picture starting at $\boxed{0}$, $\triangle$3, and using a little square like this:



These are two of Peter's small-size squares ($\frac{1}{2}$ unit on each side). Can you finish this picture?

16) When you count "over one, and up four," using the small-sized squares in question 15,

where did you end? Was it a point in The Truth set?

17) Can you describe Peter's discovery?

18) Does Peter's discovery really work?

31. Precision Drafting and Reading Numbers from Graphs

Mathematicians usually draw their graphs rather roughly and hastily, and that is usually good enough. But in today's lesson, we need to draw our graph the way a good draftsman would, very carefully and precisely.

1) The open sentence we are using is

$$\triangle = (4 \times \square) + 1.$$

Which of these graphs is the most precise?

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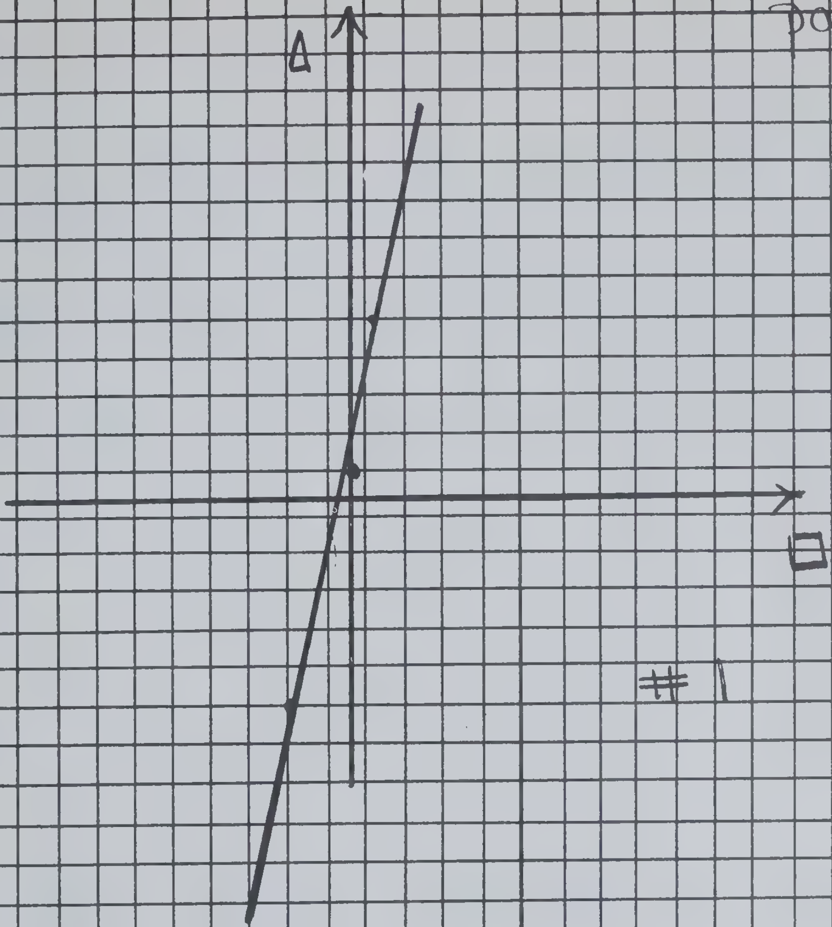
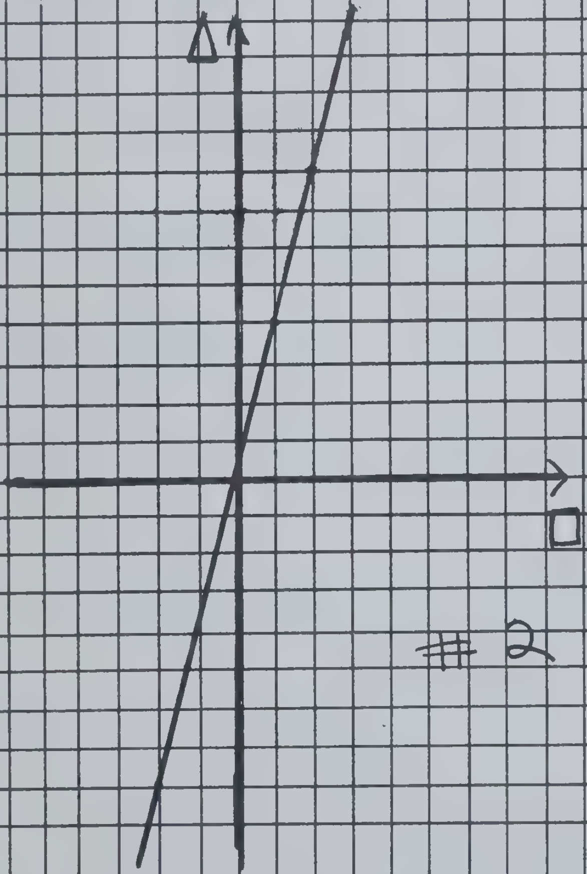
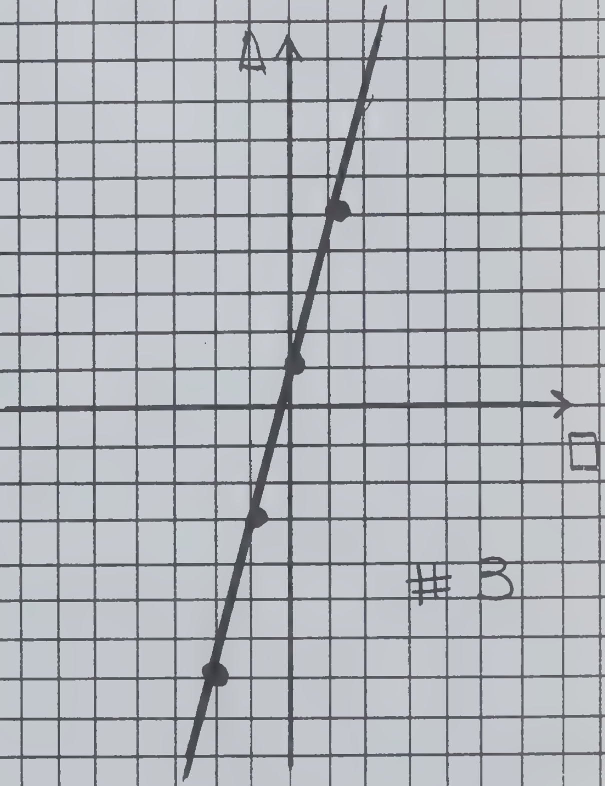


#1



#2



#3

2) Using the best of these graphs can you tell what number in the △ would correspond to 0 in the □?

Use the graph, instead of arithmetic, to answer these questions:

3) What number in the △ would correspond to 1 in the □?

4) Can you complete this table without doing arithmetic, by using the graph?

□	△
2	
3	13
-1	
-2	
$\frac{1}{2}$	

□	△
$1\frac{1}{2}$	
$-\frac{1}{2}$	
$-1\frac{1}{2}$	
	3
	4

5) Check your answer in #4 now, by substituting into the open sentence and doing the arithmetic. Were you reasonably close?

6) Can you complete this table, without doing the arithmetic, by using the graph?

□	△
$1\frac{1}{3}$	
$2\frac{1}{5}$	
$-1\frac{1}{4}$	
	7
	8
	$7\frac{1}{2}$

7) Alan says he could read the graph more accurately if it were larger. Can you find a way to make it larger?

8) Alan drew this graph for the open sentence $\triangle = (4 \times \square) + 1$.

Is it right?

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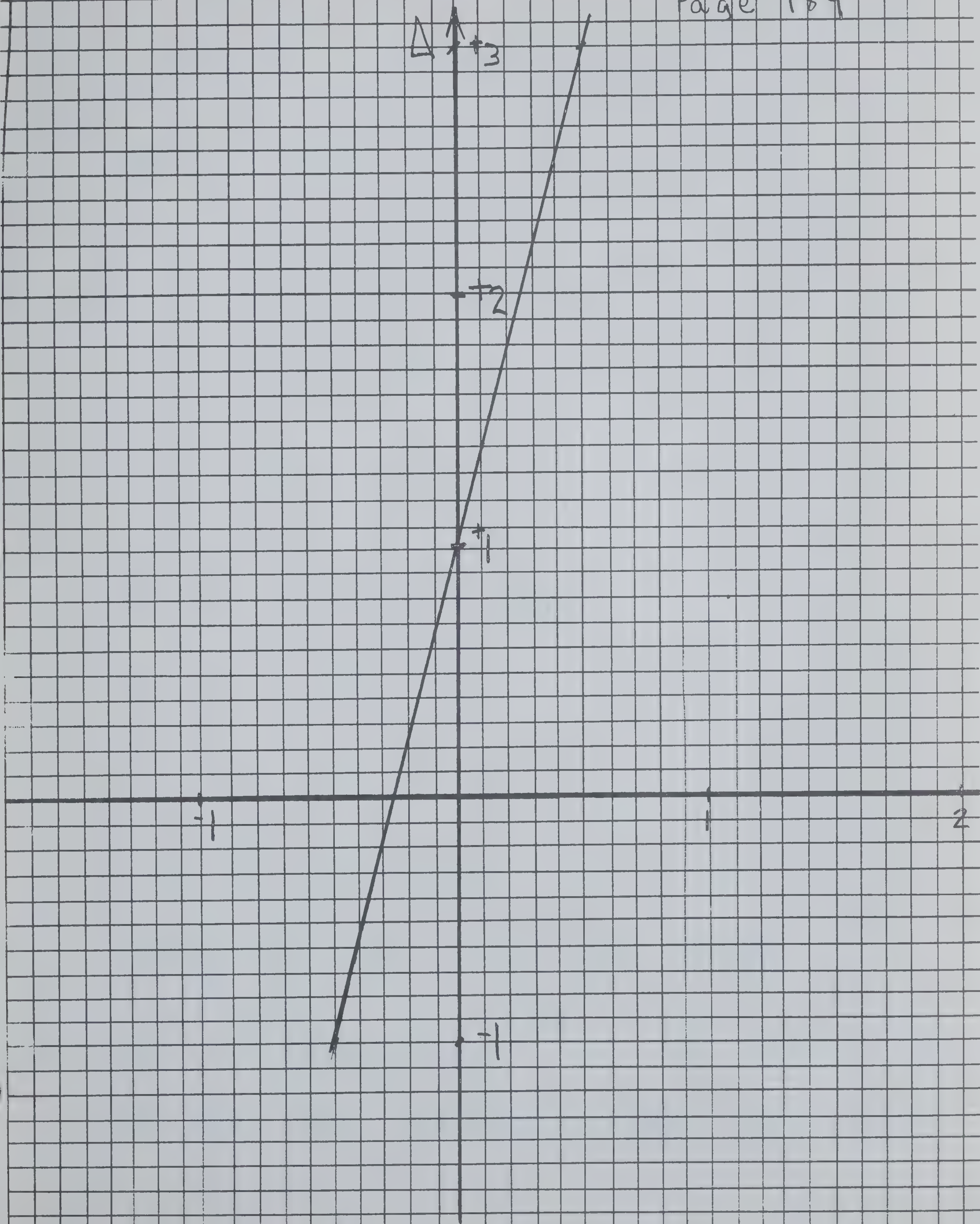


9) Using Al's "expanded" graph, can you complete this table?

□	△
1/10	
1/5	
	1 3/10
	2 1/5
-3/10	
-1/2	
1/20	

10) If you want, you can check a few of your answers to # 9 by substituting into the open sentence. Were you close?

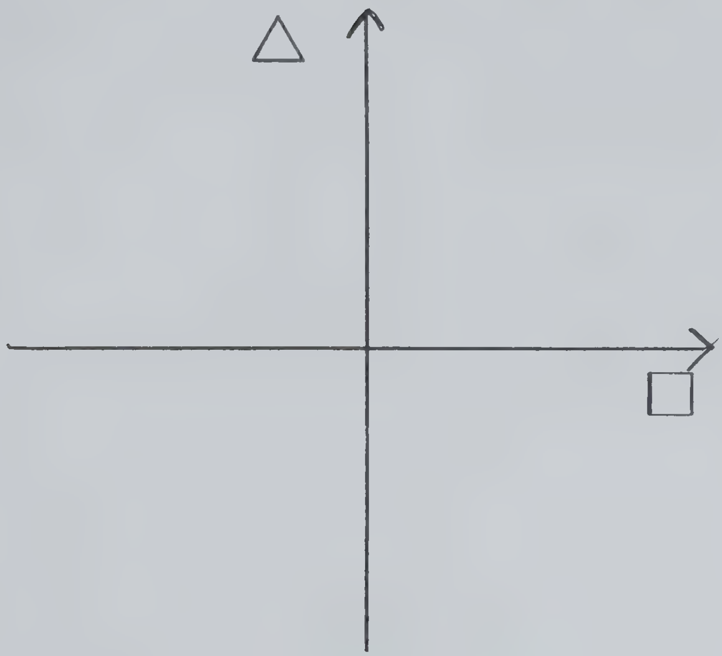
32. Letters for Placeholders

Have you looked in any of the mathematics or engineering books that are used in college? Did you find any open sentences that looked familiar?

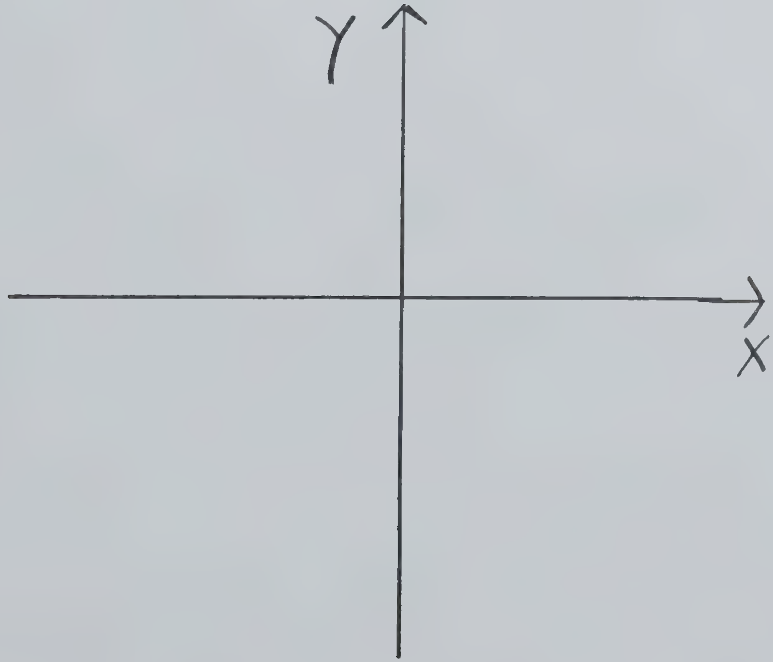
If you have looked in any of these books, you probably saw a different notation.

Where we would write □,
they would write X
Where we would write △
they would write y.

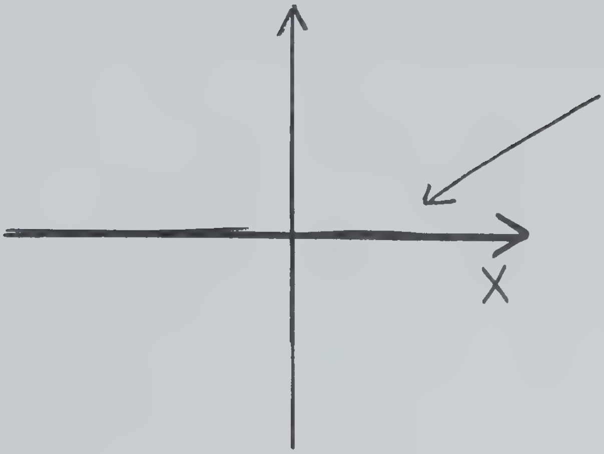
Consequently, where we would write



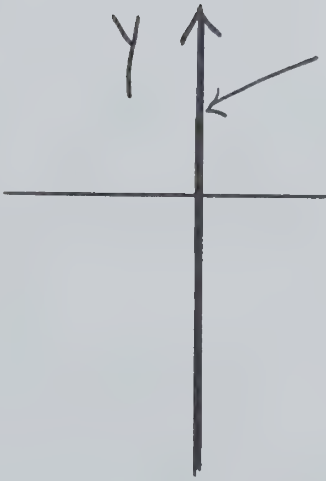
they would write



The parts of a graph are given special names:

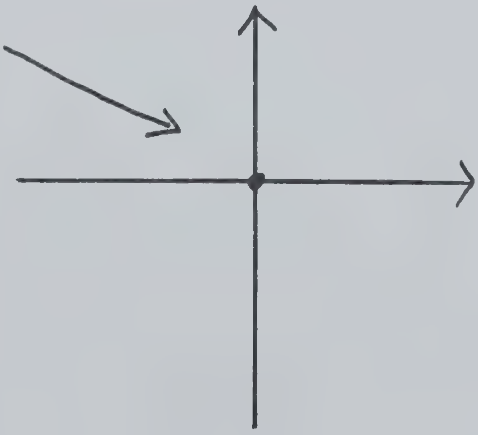


This is called the "X - axis"

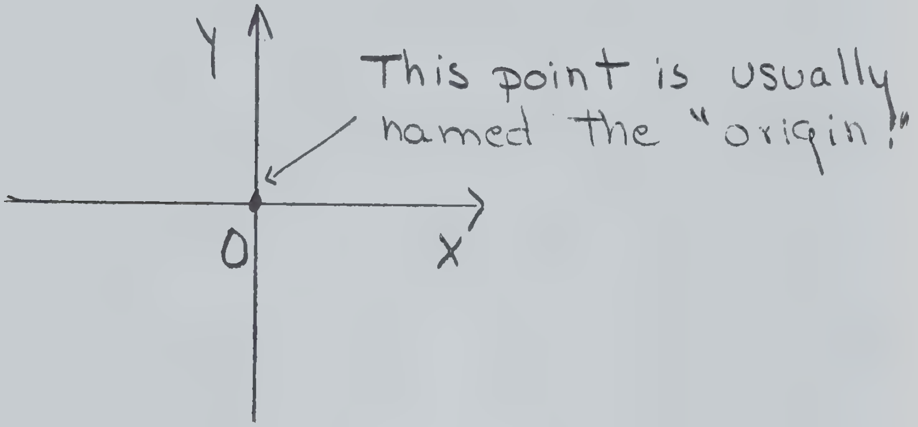


This is called the "Y - axis"

The point where the x-axis Crosses the y- axis



is usually labelled with a capital letter "O," and is called the "origin."



1) Where we would write

$$\triangle = \square + 3,$$

what would most college books write?

2) Jerry found this in a college book:

$$y = x + 7.$$

Can you write this using "$\square$" and "$\triangle$" instead of "x" and "y"?

3) Wayne says that we will have trouble with the "x" sign for multiplication, because people will confuse it with the letter "x" for "□".

Do you know how college books write "2 times x" [meaning "$2 \times \square$"]?

4) Jerry found these in some college books. Can you re-write them, using "□" and "△" notation?

a) $y = (4x) + 3$

b) $y = (3x) - 2$

c) $x + y = 15$

d) $y = x - 5$

e) $y = 2x + 1$

f) $3y + 4x - 2 = 0$

5) Did you know a shorter way of writing $x \cdot x$?

6) Can you write $y = x \cdot x$ in "□", "△" notation ?

7) What is the value of

a) $3^2 =$

b) $5^2 =$

c) $7^2 =$

d) $11^2 =$

8) Can you write this in "□", "△" notation ?

$$x^2 + y^2 = 25$$

9) Can you make a graph to show the truth set of $x^2 + y^2 = 25$?

10) How would you describe the graph of the truth set for $x^2 + y^2 = 25$?

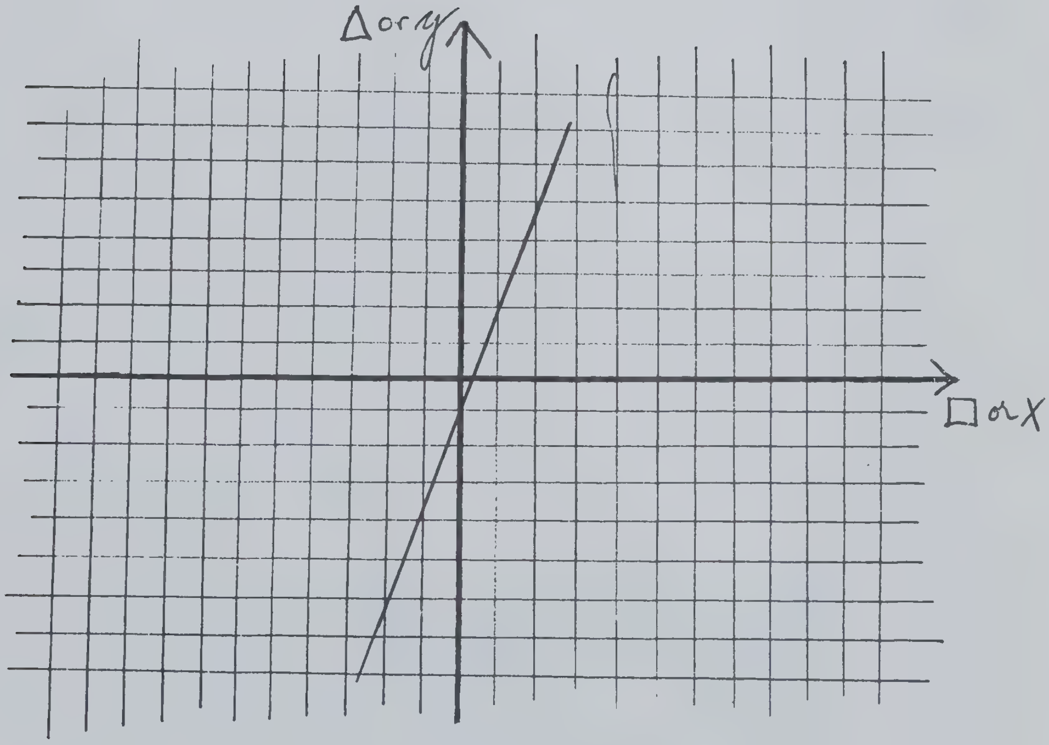
11) If you want, you can make a big graph of $x^2 + y^2 = 25$, by using 5 squares for each unit. Can you read off some fractions from this graph?

12) Did you check these fractions by substituting into the open sentence? Were you close?

33. Describing Graphs of Truth Sets

1) AL: Dick, draw this line. The pattern is "over one to the right, and up three."

Dick: OK – Here is the line;

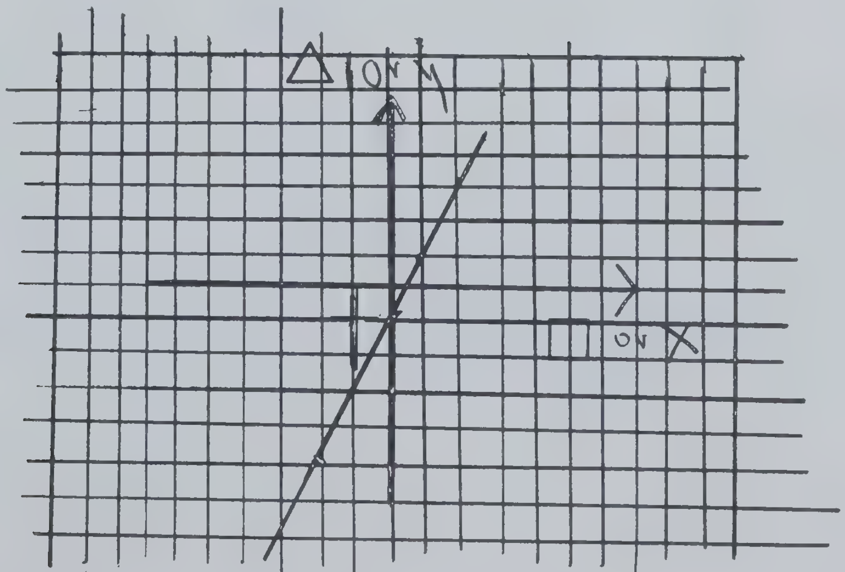


Is Dick's line the one that *you* would have drawn?

2) Al: No! That's not the line I mean! The line I meant crosses the y- axis at ⁺2. Can you draw the line that Al meant?

3) Jerry: Bill, draw the line that crosses the y-axis at -1, and has the pattern "over one to the right, and up two".

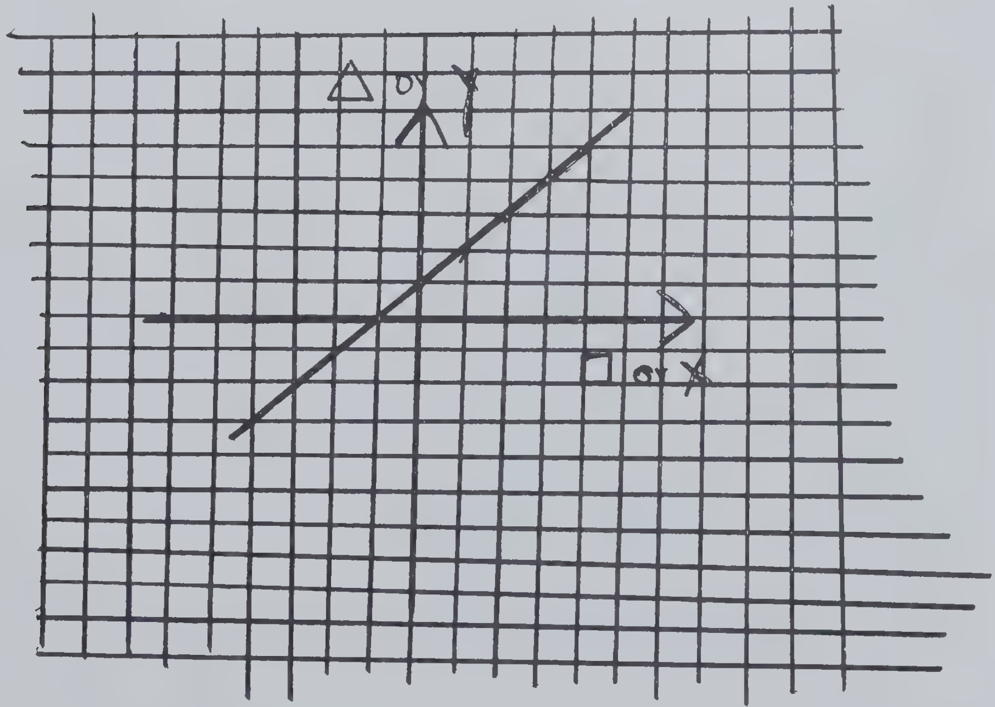
Bill: O.K. You mean this line here, don't you?



Was Bill's line right?

4) Paul: Ellen, will you draw the line that crosses the X-axis at -1, and has the pattern "over one to the right, and up one."

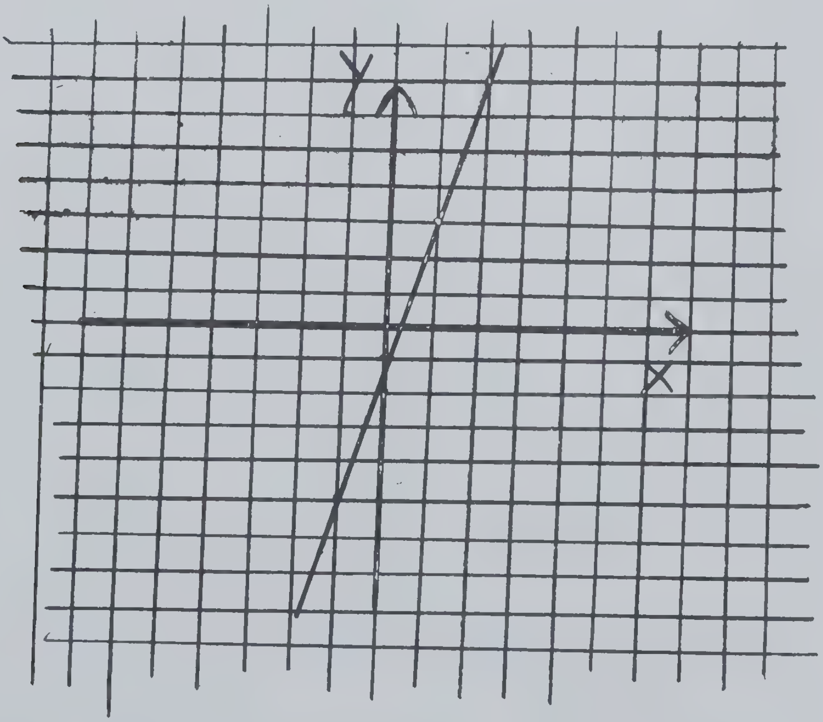
Ellen: Just a minute ... yes, here is the line:



Do you think Ellen's line is the one that Paul described?

5) Mary: Andy, will you draw a line that crosses the y-axis at -3, and has a pattern "over to the right, and up four?"

Andy: O.K... Here is the line:



Do you think that Andy's line is the one that Mary asked for?

Can you draw these lines:

6) A line that crosses the x-axis at $^{-}3$, and has a pattern "over one to the right, and up one"?

7) A line that crosses the y-axis at $^{+}5$, and has a pattern "over one to the right, and down one"?

8) A line that crosses the y-axis at $+6$, and crosses the x-axis at $^{-}2$?

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9) A line That crosses the y-axis at $^{+}5$, and has the pattern "over 3 to the right and down 1"?

10) A line that has the pattern "over 1 to the right, and up 3"?

11) A line that crosses the y-axis at $^{+}8$, and has the pattern "over 1 to the right, and down zero"?

34. Myra's Discoveries

1) Can you draw a straight-line graph with the pattern "over one to the right, and up two"?

2) Can you make a table for the truth set of this open sentence:

$$\triangle = (3 \times \square) + 2 \text{ ?}$$

3) For this same open sentence,

$$\triangle = (3 \times \square) + 2,$$

can you represent the truth set by a graph?

4) Myra says that she has discovered two secrets. She can draw the graph of an open sentence like

$$\triangle = (3 \times \square) + 4$$

just by looking at the equation.

Do you know Myra's two secrets?

(If you do, don't tell!)

5) Can you draw a line that crosses the y-axis at $^{+}2$, and has a pattern "over one and up three"?

Can you graph the truth set for each open sentence?

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6) $\triangle = ({}^{+}3 \times \square) + {}^{-}2$

7) $\triangle = ({}^{+}2 \times \square) + {}^{+}5$

8) $\triangle = ({}^{+}1 \times \square) + {}^{+}8$

9) $\triangle = ({}^{+}1 \times \square) + {}^{-}5$

10) $\triangle = ({}^{-}2 \times \square) + {}^{+}3$

11) $\triangle = ({}^{+}\tfrac{1}{2} \times \square) + {}^{+}4$

12) $\triangle = ({}^{-}\tfrac{1}{2}\ \square) + {}^{+}3$

13) $\triangle = ({}^{+}\tfrac{1}{4} \times \square) + {}^{-}6$

14) $\triangle = ({}^{+}1 \times \square) + 0$

15) $\triangle = (0 \times \square) + {}^{+}5$

16) $\triangle = ({}^{+}5 \times \square) + {}^{-}7$

17) $\triangle = (-\tfrac{1}{4} \times \square) + {}^{+}6$

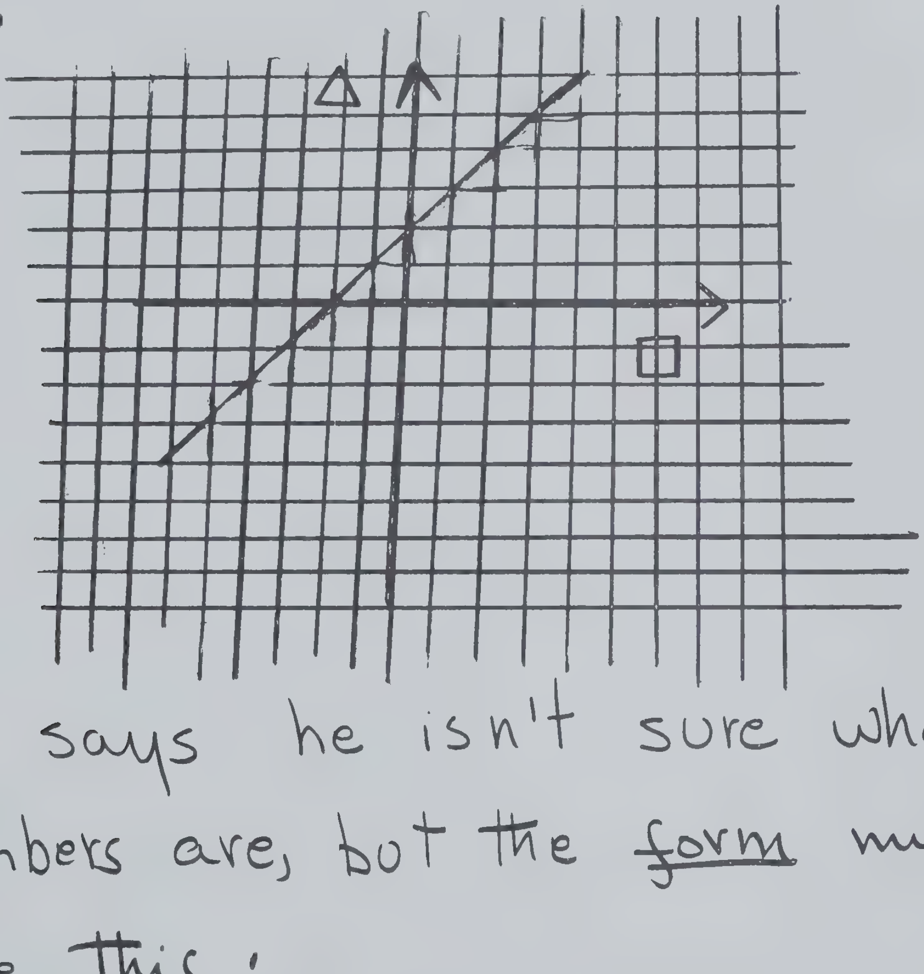
18) $\triangle = ({}^{+}5 \times \square) + {}^{+}1$

19) Have you discovered Myra's two secrets.

Remember - If you have, don't tell!

35. Writing Equations For Graphs

1) Can you write the equation for this graph?



2) Paul says he isn't sure what the numbers are, but the form must be like this:

$$\triangle = (\_\_\_ \times \square) + \_\_\_$$

some number here

some number here

Do you agree with Paul?

3) John, who uses x's and y's, says in his notation the form would look like this:

$$y = \left(\underset{\uparrow \atop \text{some number here}}{\underline{\qquad}} \cdot x\right) + \underset{\uparrow \atop \text{some number here}}{\underline{\qquad\qquad}}$$

Is John right?

4) Mary says the equation should be

$$\triangle = ({}^{+}1 \times \square) + {}^{+}2.$$

Do you agree?

5) Jerry says he thinks the equation should be

$$\triangle = ({}^{+}2 \times \square) + {}^{+}1.$$

Is Jerry right?

6) Carl says that if you use Jerry's equation, you get this table for the truth set:

□	△
0	$^{+}1$
1	3
2	5
3	7
$^{-}1$	$^{-}1$
$^{-}2$	$^{-}3$

Table for the Truth Set for Jerry's equation

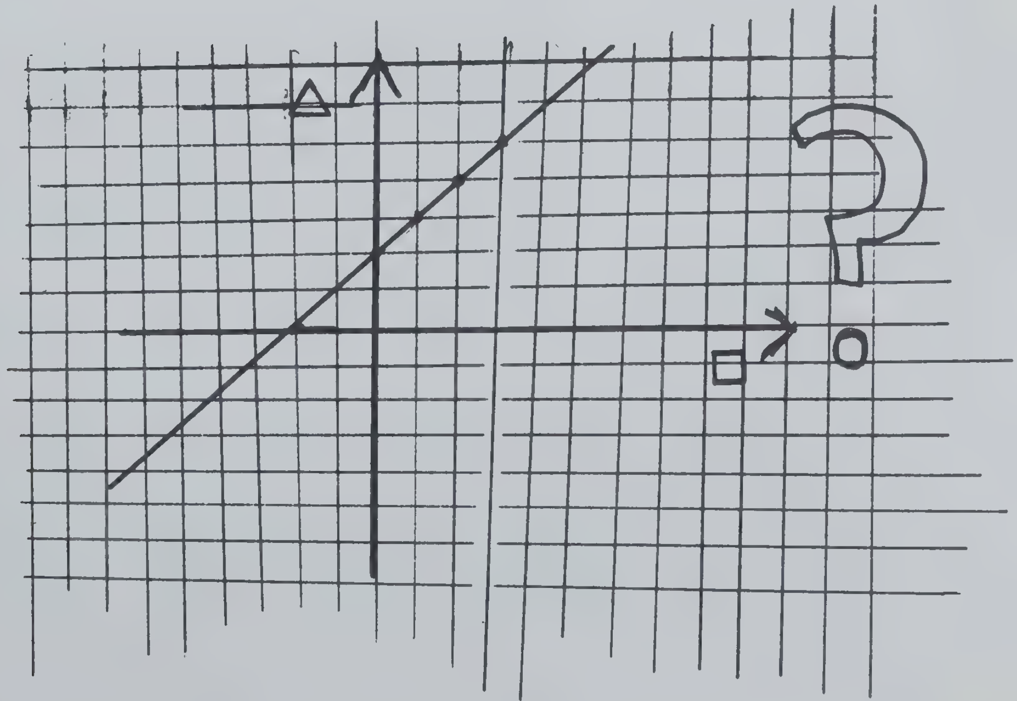
Is Carl right?

7) Marie says that if you use Mary's equation, you get this table for the truth set:

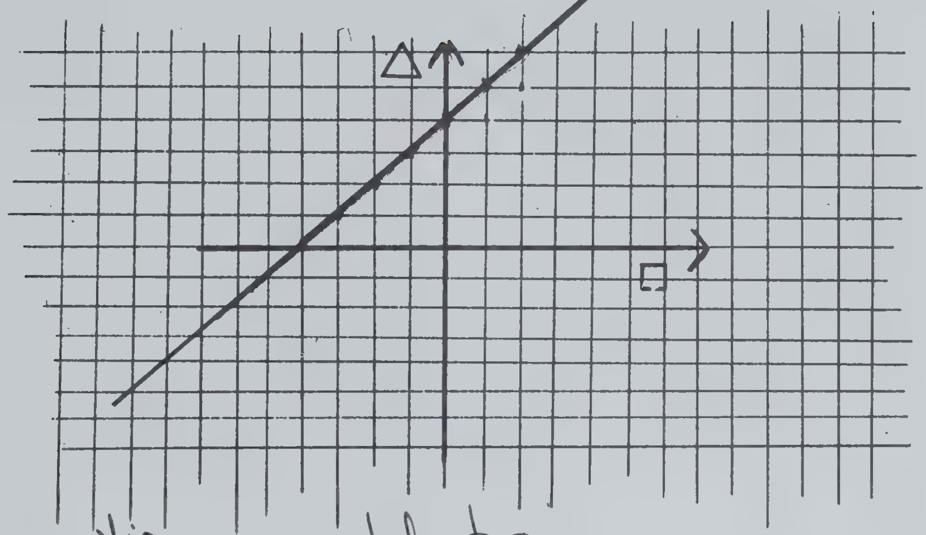
□	△
0	2
1	3
2	4
-1	1
-2	0
-3	-1
-4	-2

Is Marie right?

8) Did either Jerry or Mary have the right equation for the graph:



9) Wayne says that for this graph,



the equation would be

$$\triangle = ({}^{+}1 \times \square) + {}^{+}4,$$

which could also be written

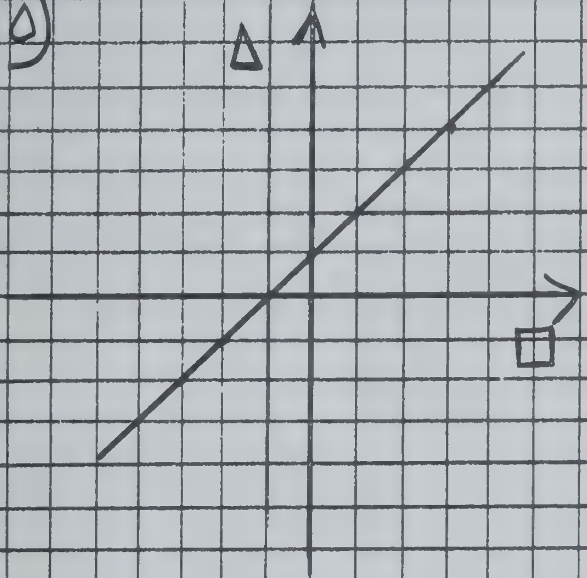
$$y = ({}^{+}1 \cdot x) + {}^{+}4.$$

Is Wayne right?

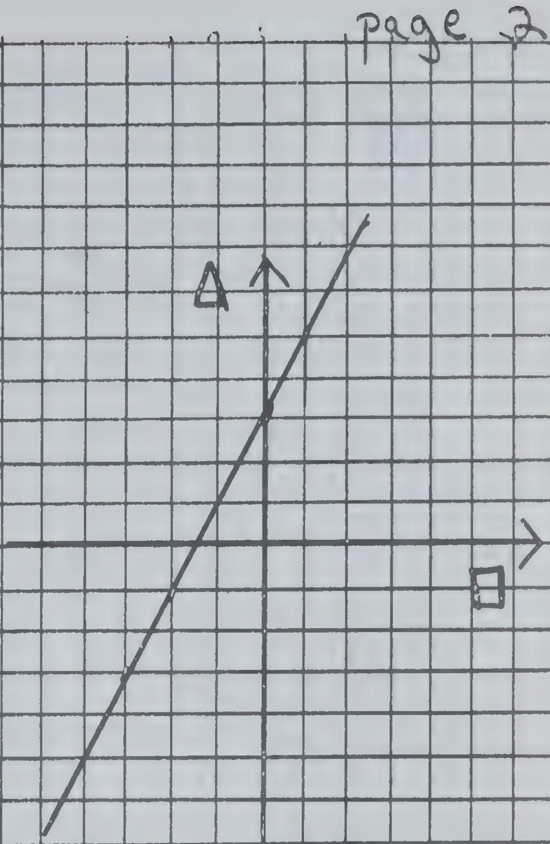
Can you write the equation for each of

These graphs —

10)



11)



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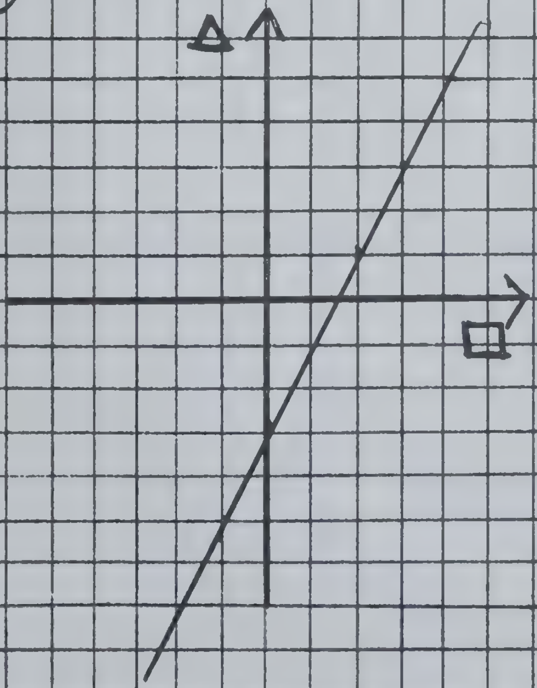

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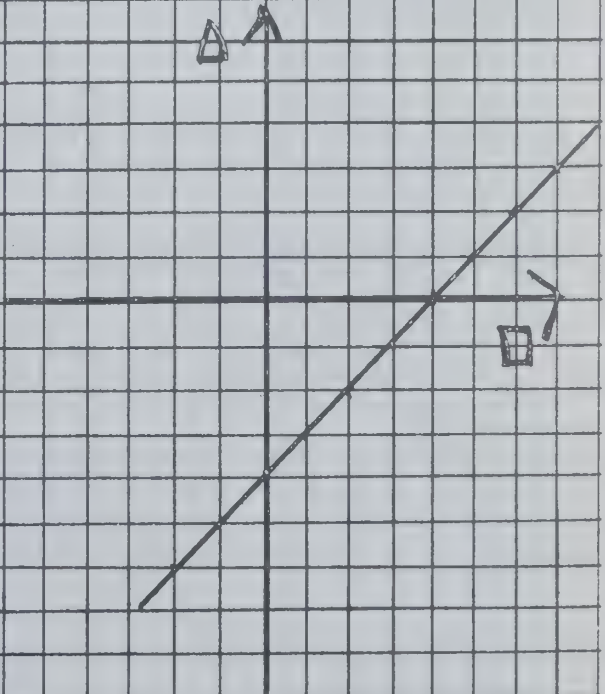

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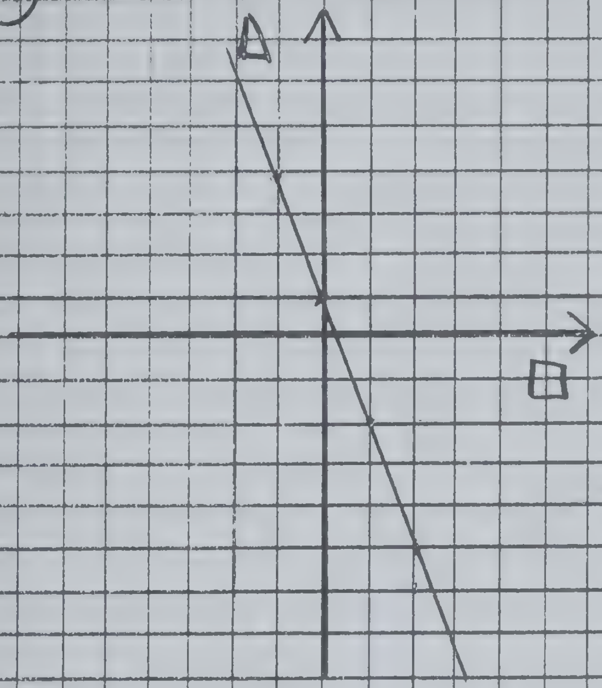
14)



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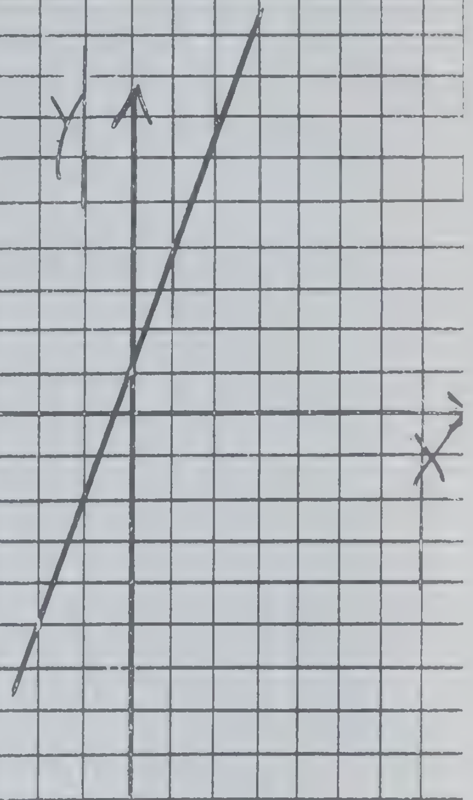

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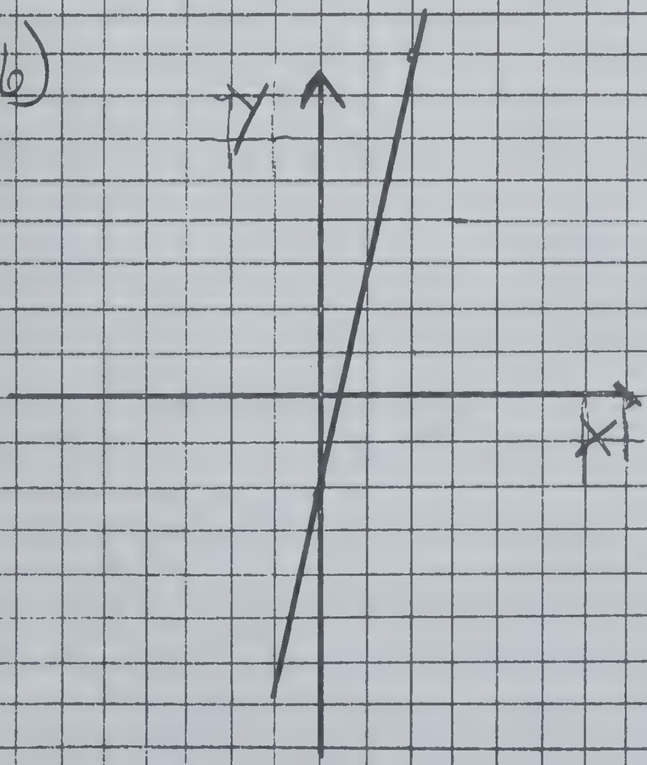

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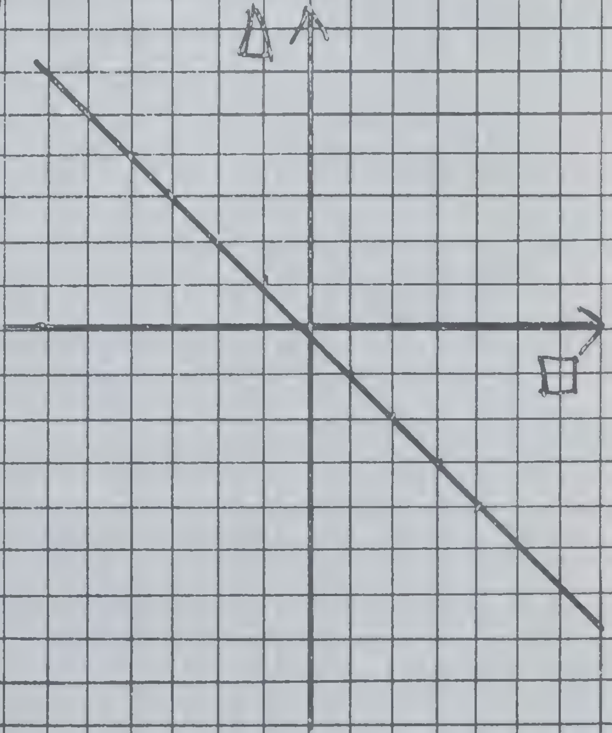

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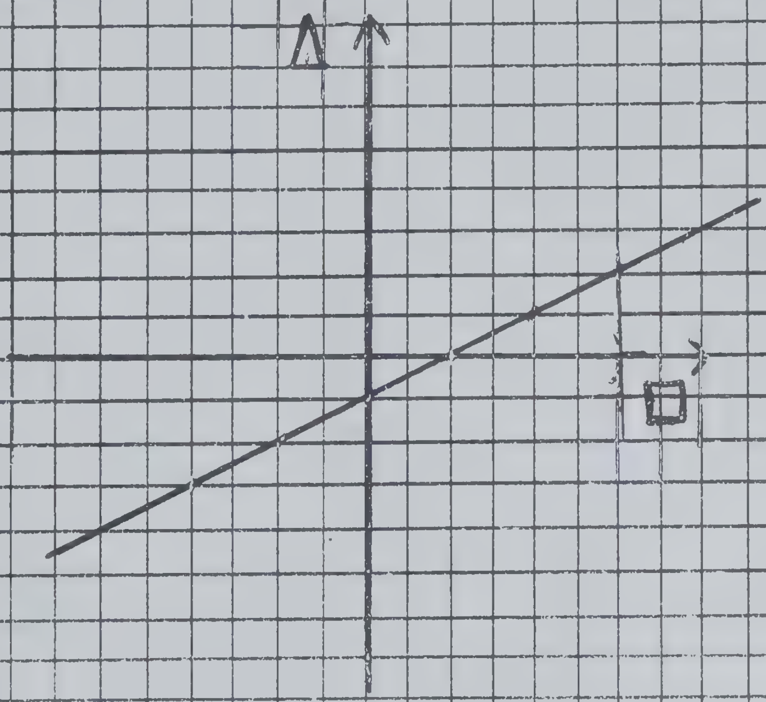

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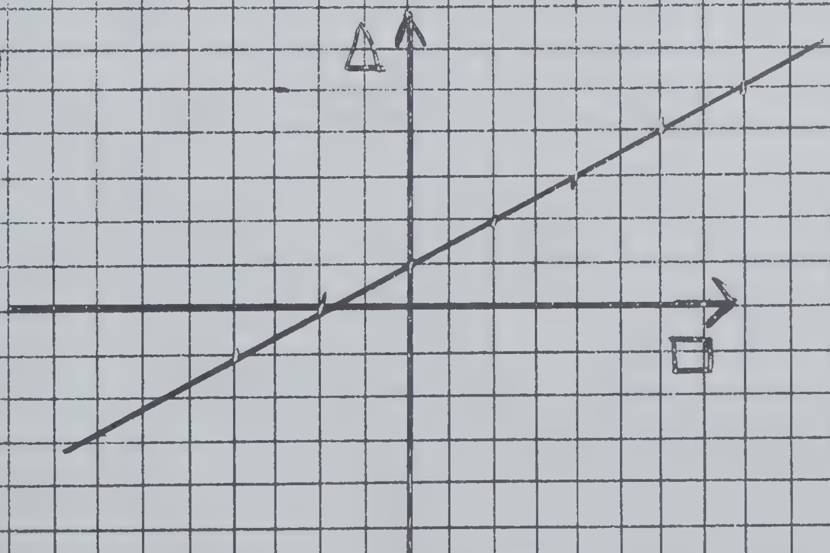


20) Jerry says that the "pattern" in the problem 19) is "over two to the right, and up one."

Is Jerry right?

21) Kathy says that the pattern in question 19) is "over one to the right, and up ½."

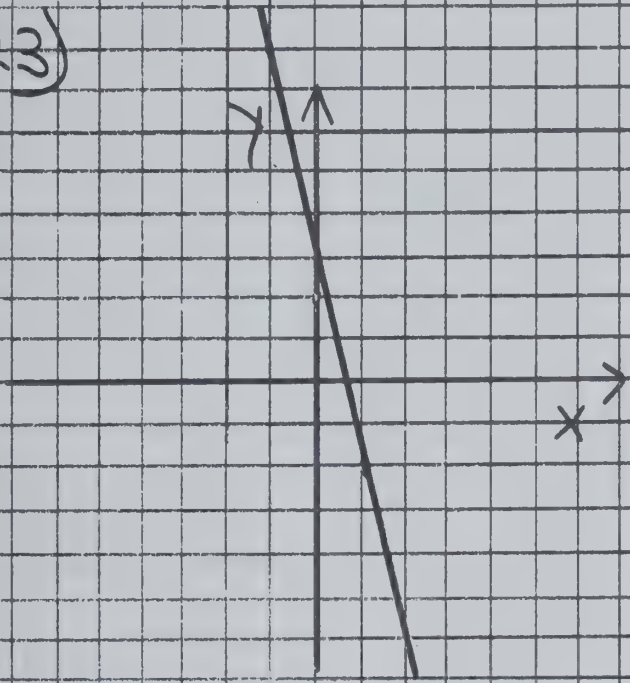
Is Kathy right?

22) Al says that Kathy and Jerry are both right.

Do you agree?

Can you find the equations for these graphs?

23)



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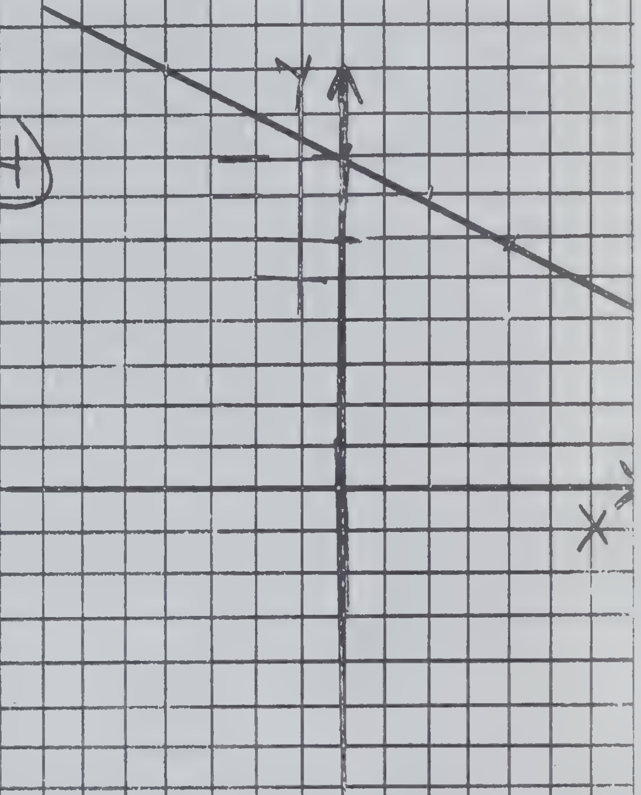

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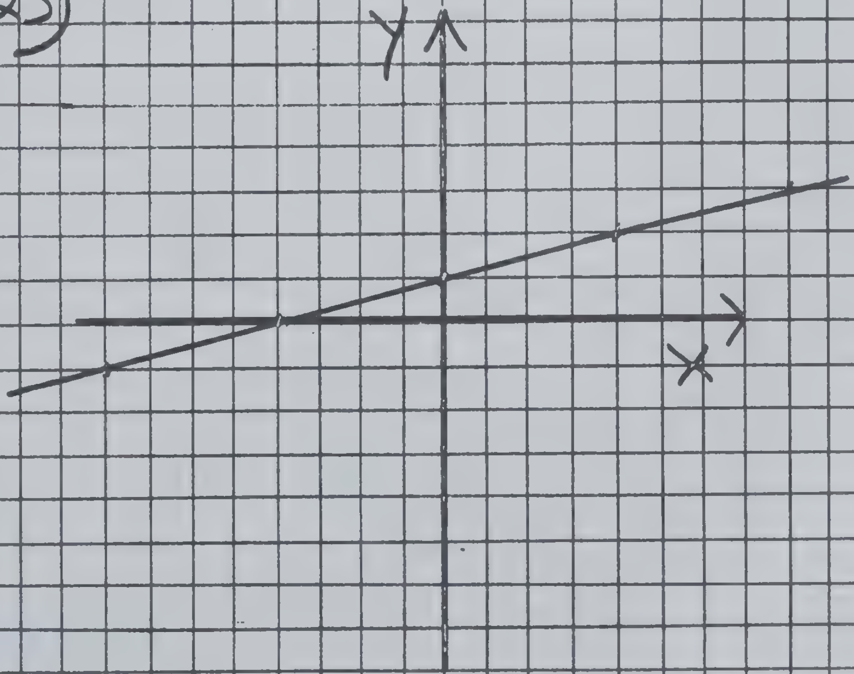

26) Jerry says the pattern in question 25) is "over four to the right, and up one." Do you agree?

27) Ellen says the pattern is "over two to the right, and up ½." Is Ellen right?

28) How else can you describe the pattern in question 25)?

Can you find the equations for these graphs?

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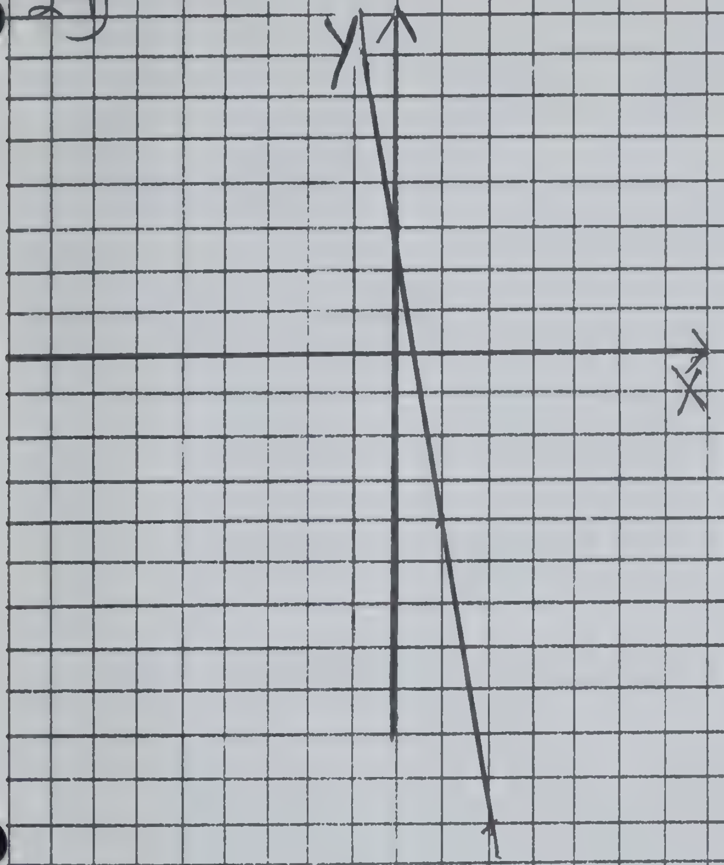
29)



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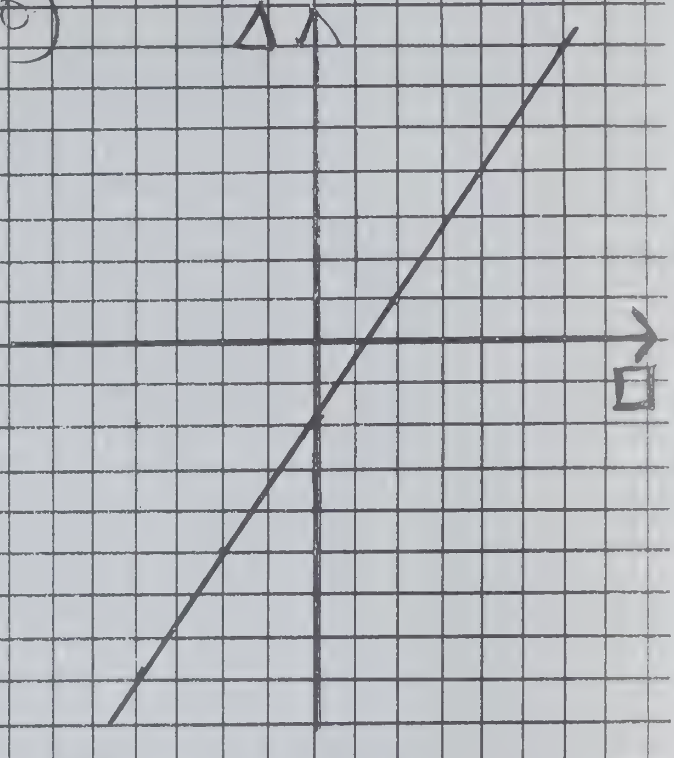

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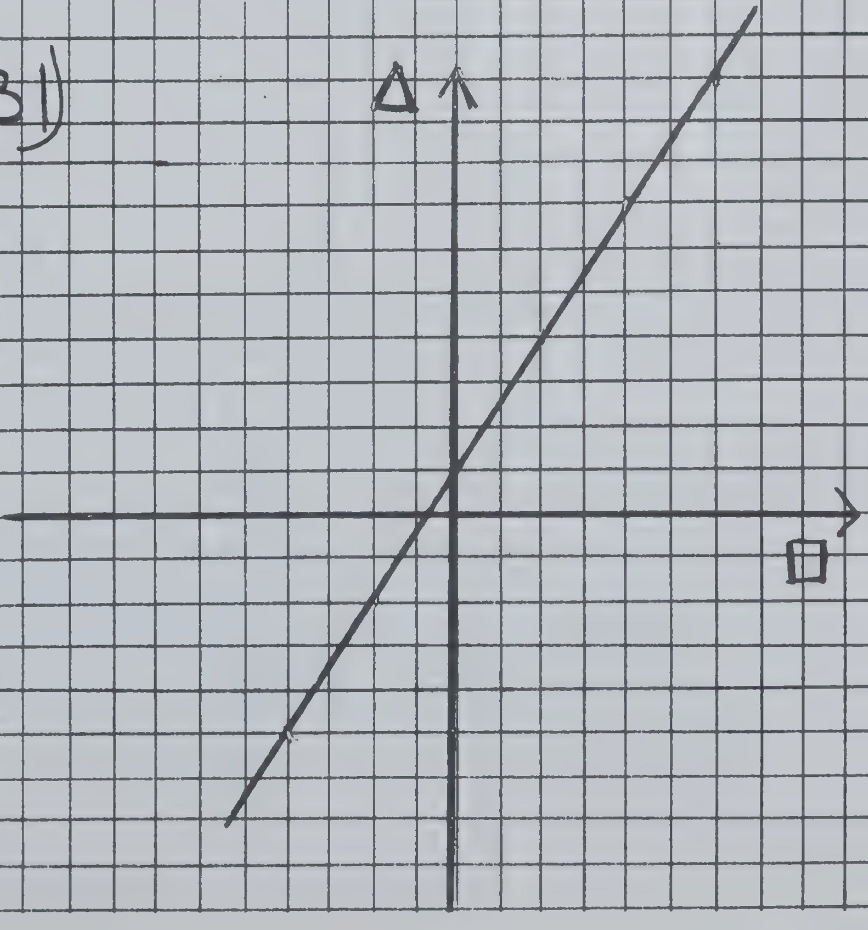
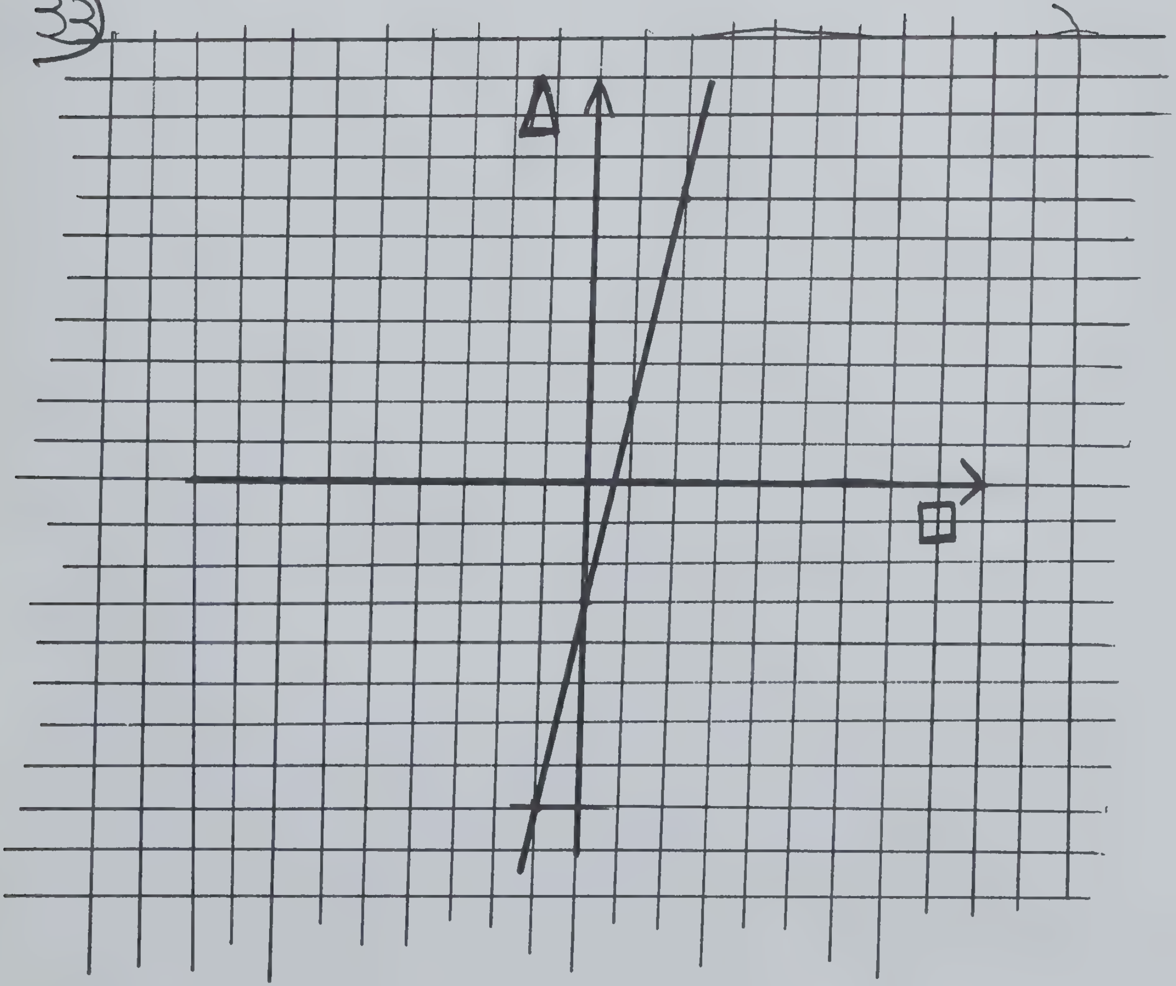


32) How would you describe the "pattern" in question 31? Is there any other way to describe this pattern?

Can you find the equations for these graphs?

33)



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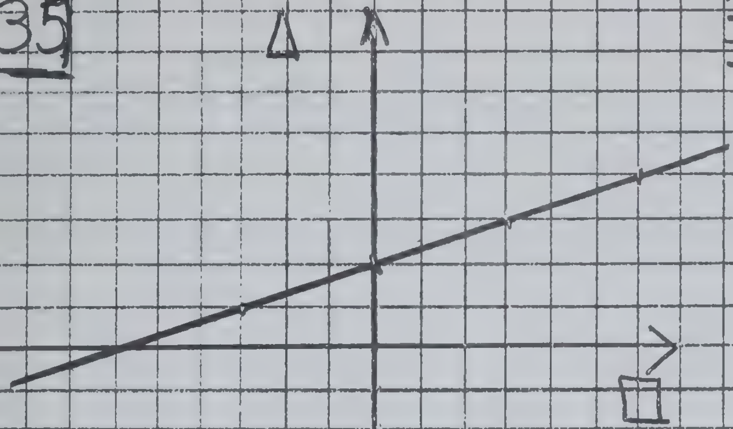
35)



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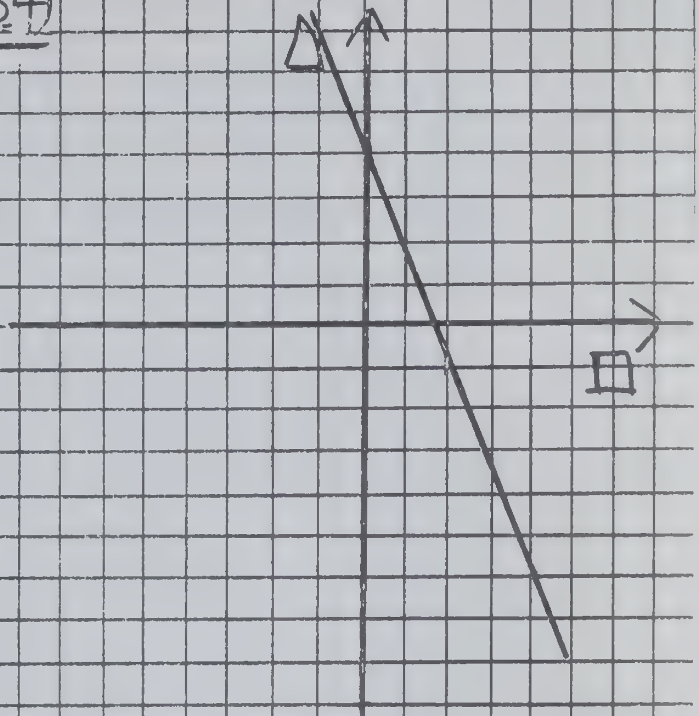

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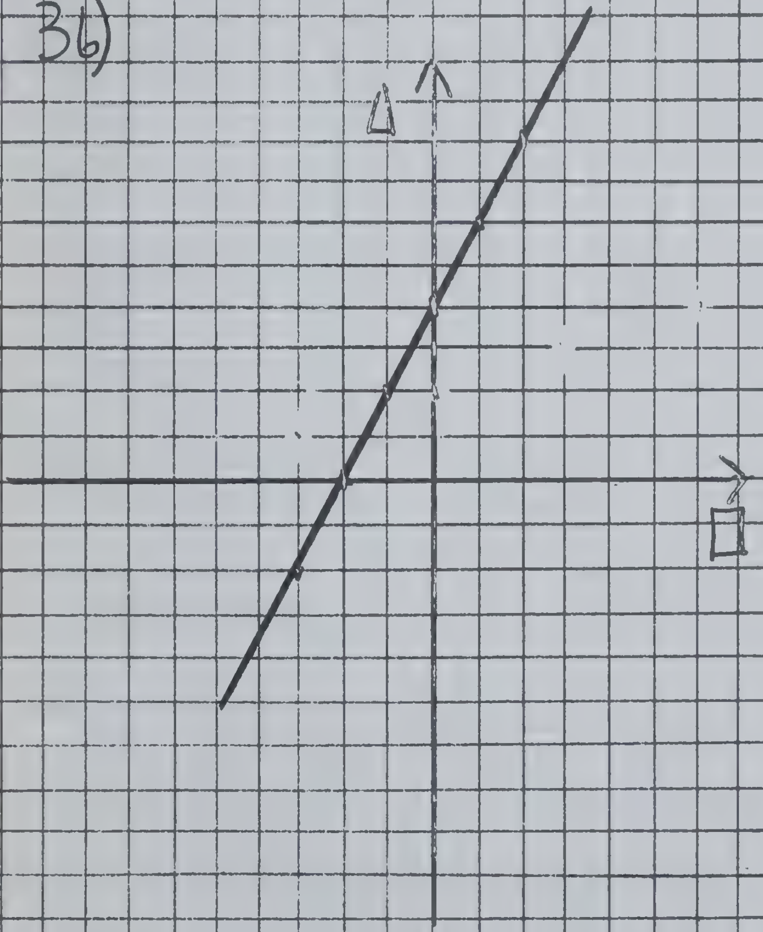

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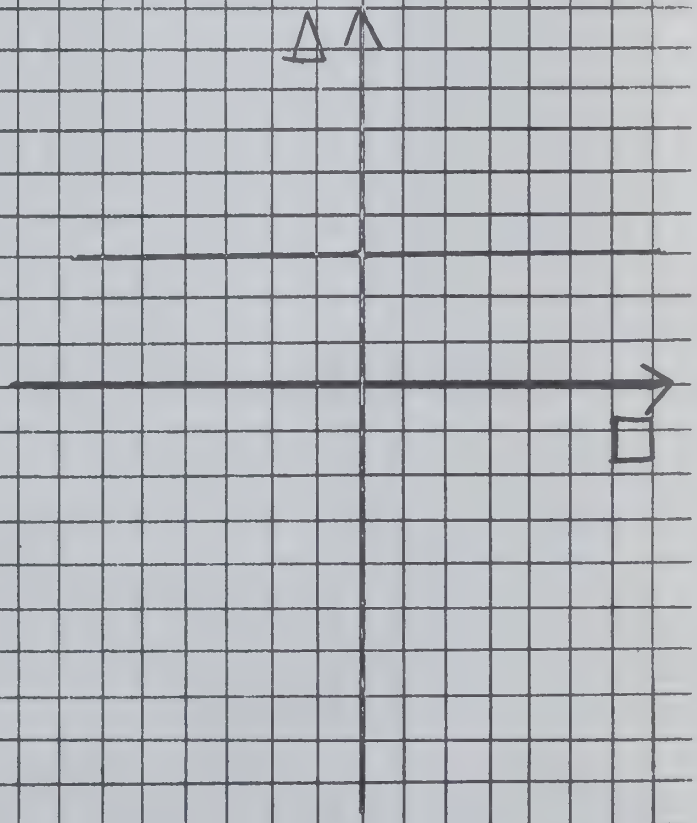
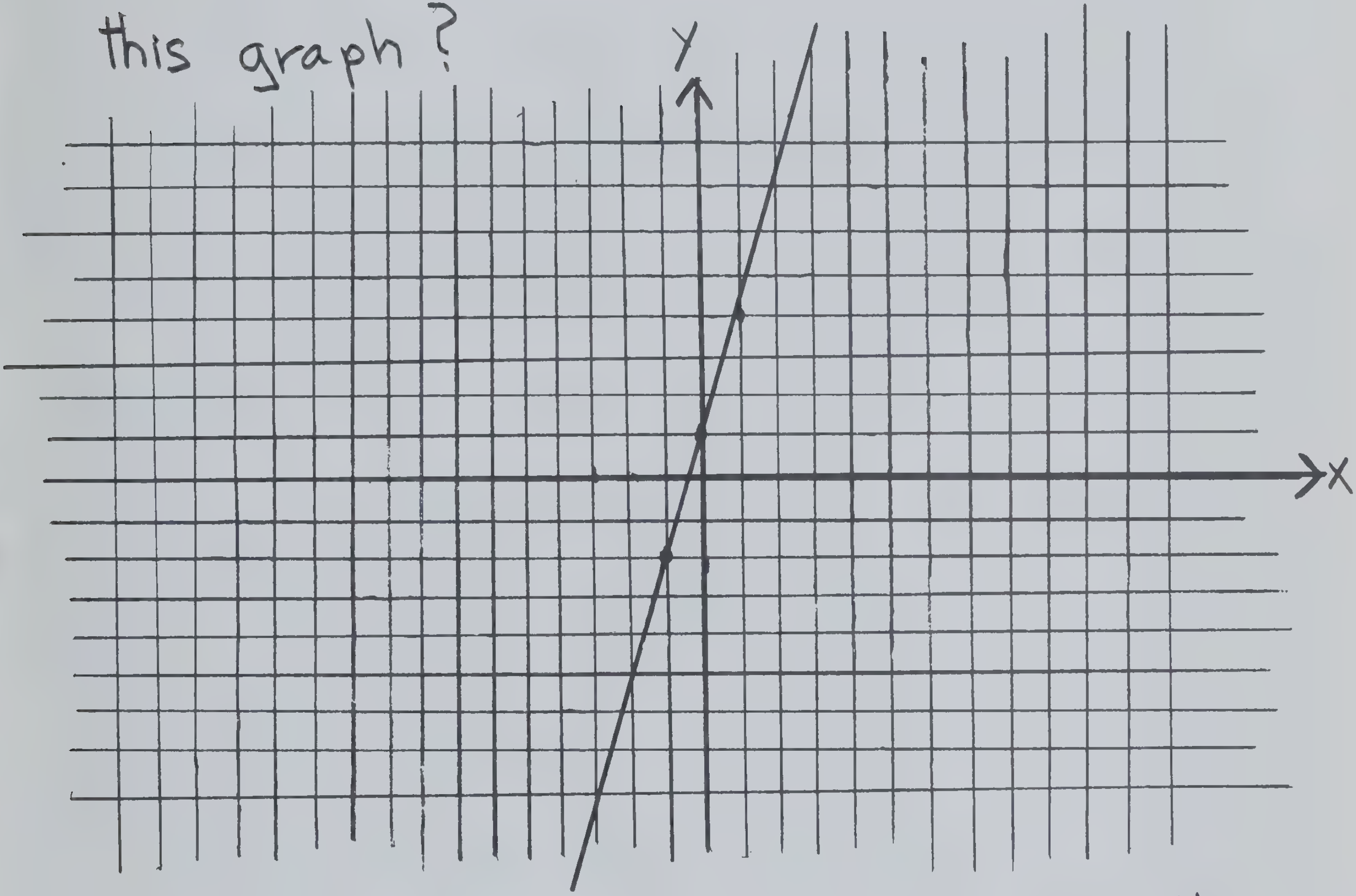


36. The Equation of a Graph

1) Can you write the equation for this graph?



2) Judy says it must be of this form:

$$\triangle = (\_\_\_ \times \square) + \_\_\_$$

some number here

some number here

Do you agree?

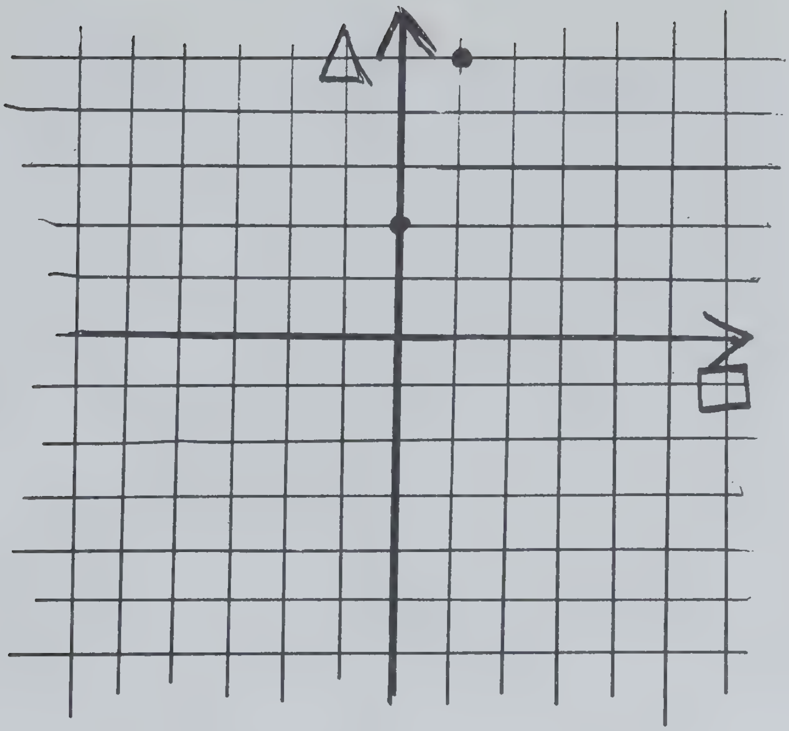
3) Doris says the equation is

$$\triangle = (3 \times \square) + 1.$$

Do you agree?

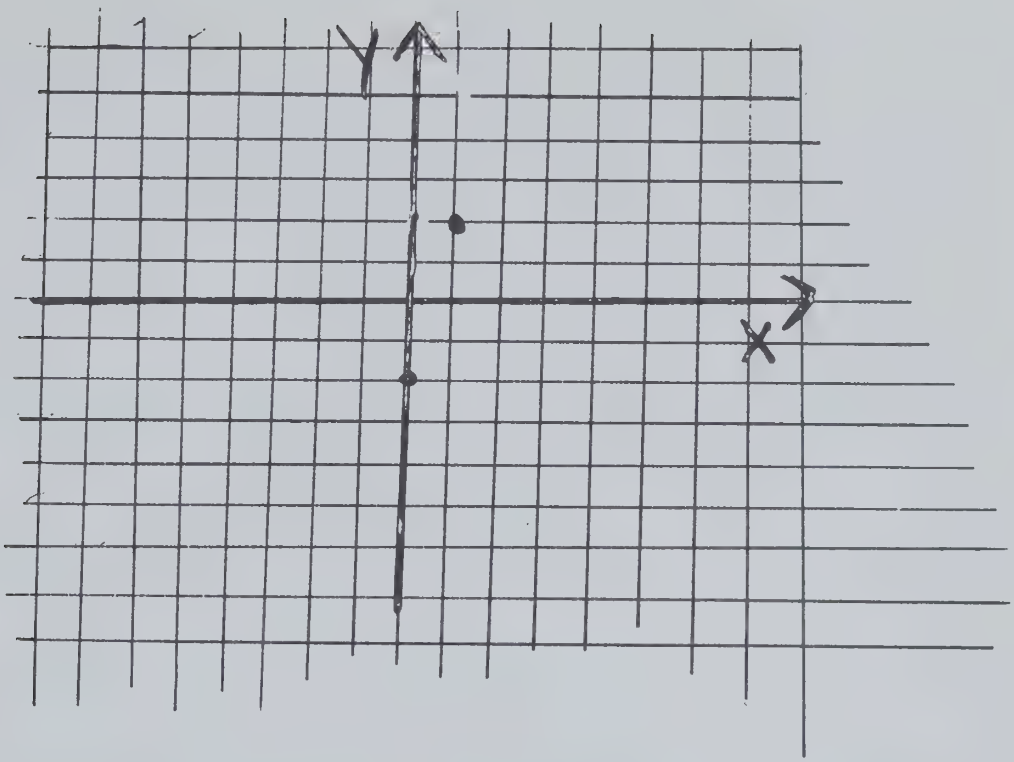
4) Bill says that all he needs are two points on a straight line graph. Then he can fill in as many other points as he wants, and he can write the equation for the graph.

Can you fill in some more points on this straight line graph?



5) Can you write the equation for the graph in question 4)?

6) Can you fill in some more points on this straight line graph?

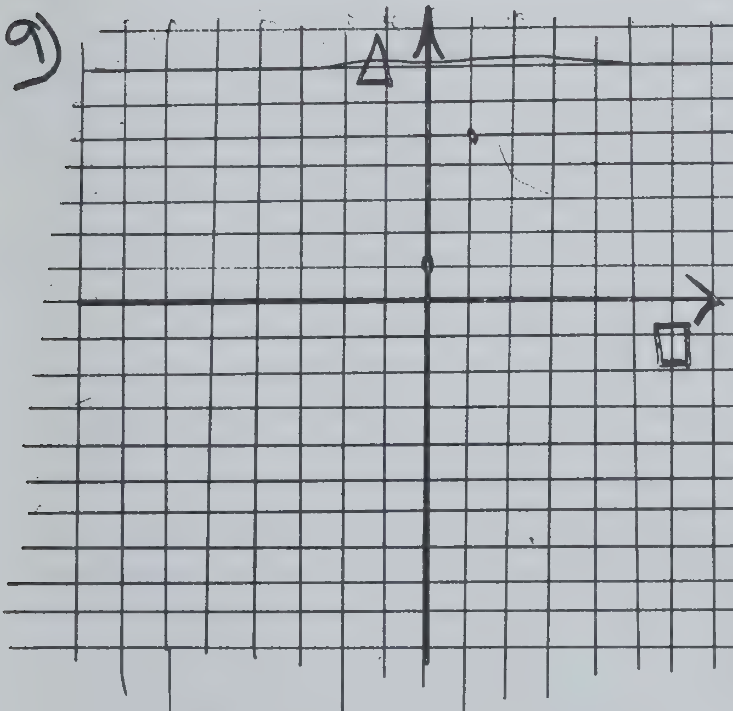


7) Can you write the equation for the graph in question 6) ?

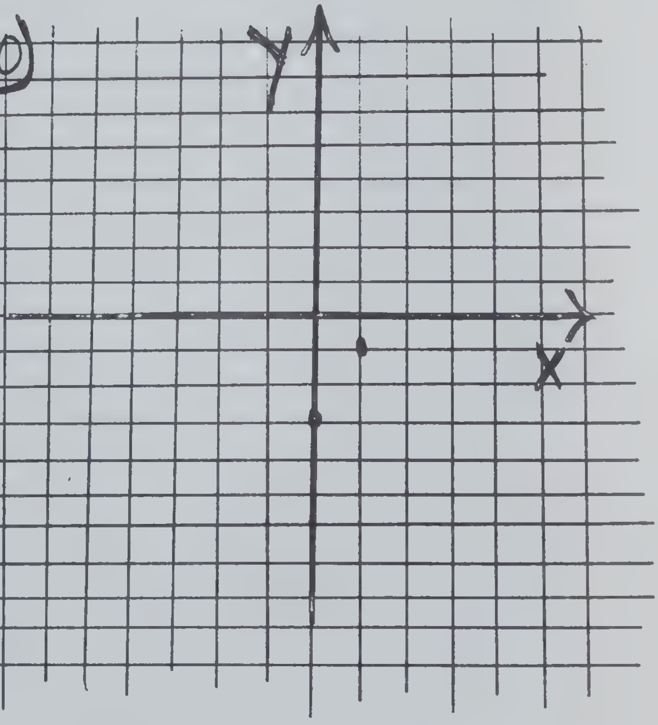
8) Is Bill right in saying that you only need to tell him two points on a straight line graph, and after that he can figure out the others by himself ?

For each of these straight line graphs, can you draw the graph? Can you find the right equation?

9)



10)



△ = (___ × □) + ___

y = (___ · X) + ___

△ = (___ × □) + ___

y = (___ · X) + ___

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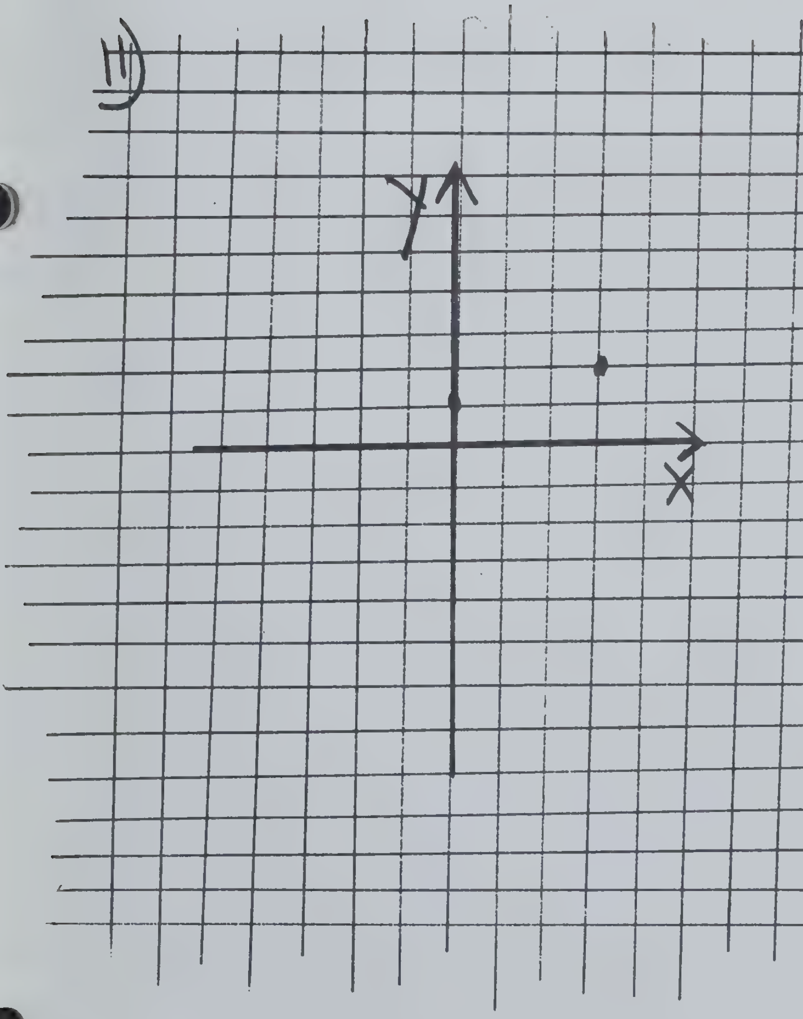
11)



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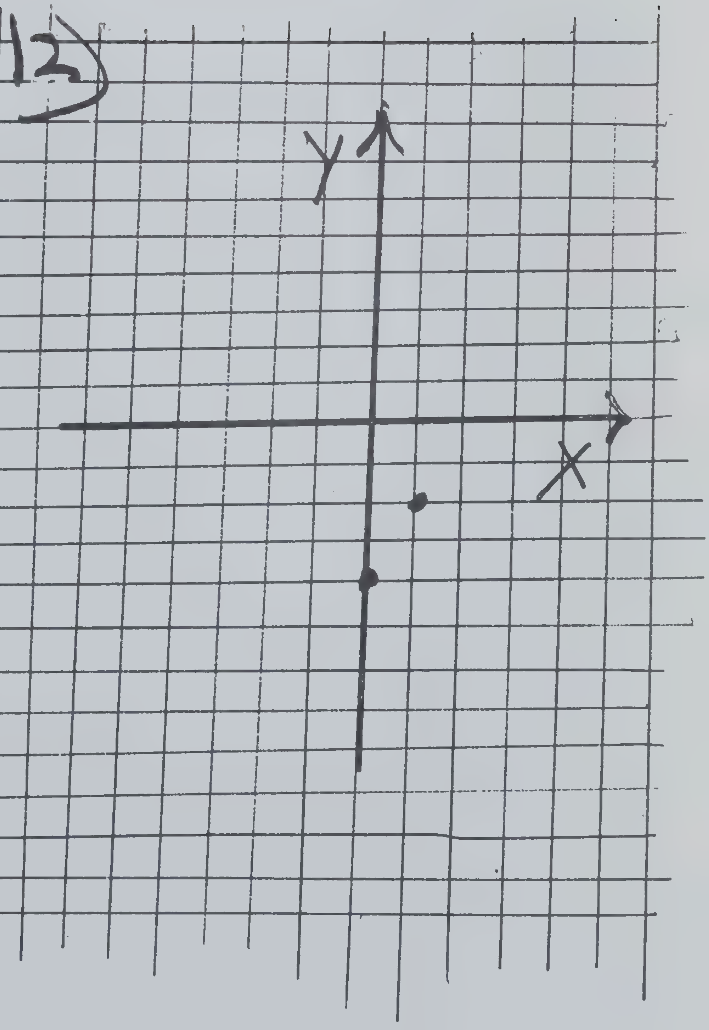

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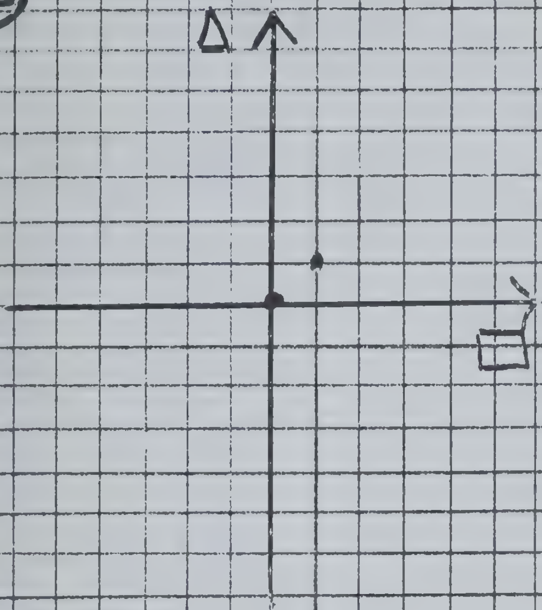

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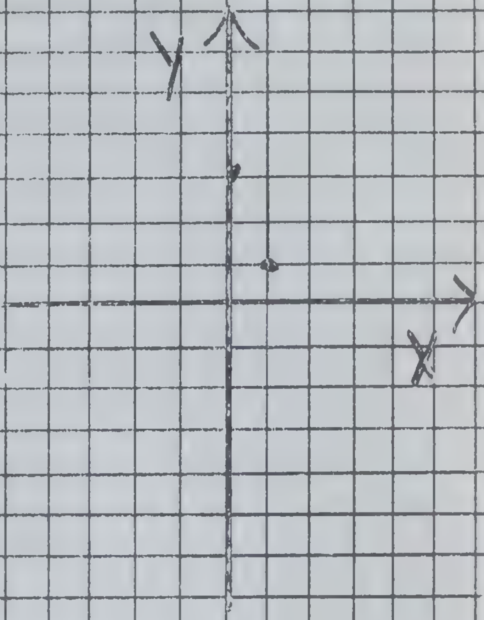

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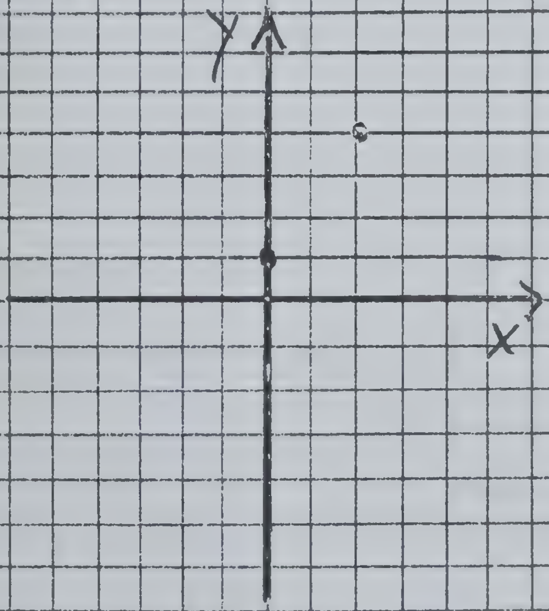

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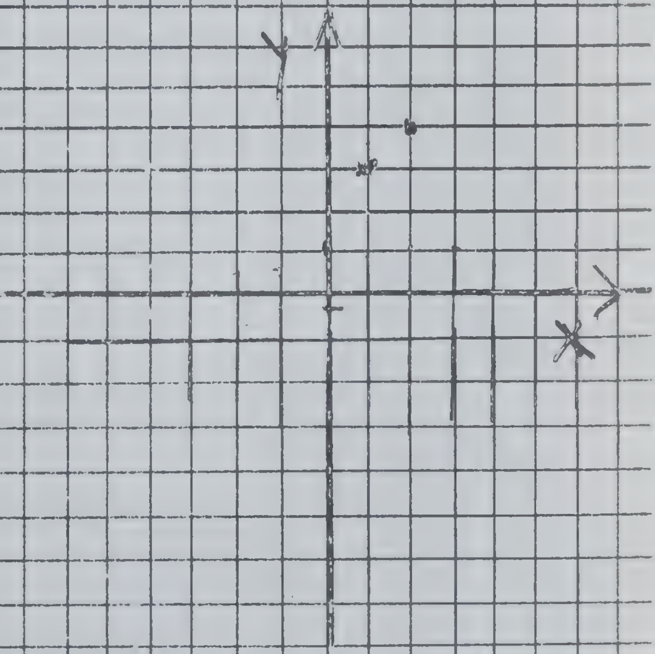

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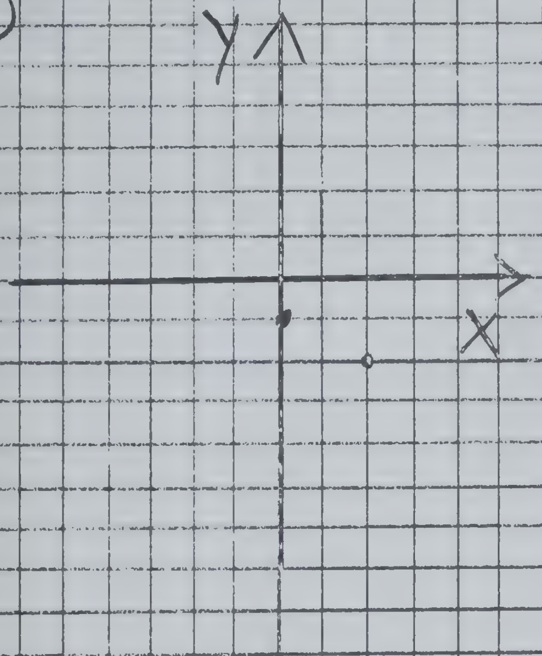
17)



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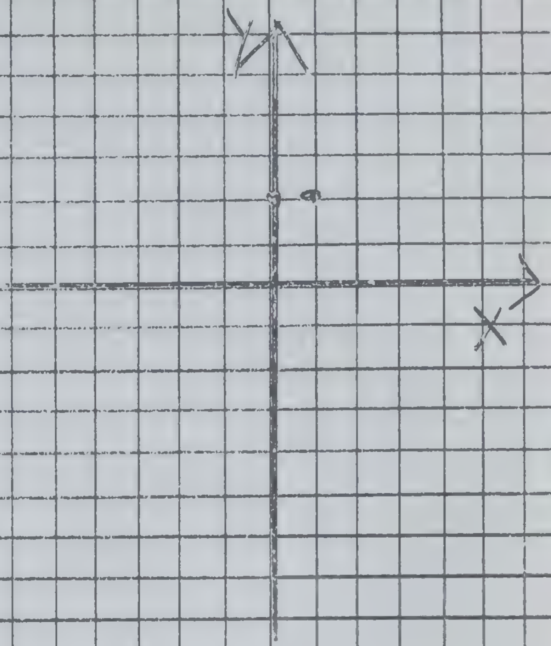

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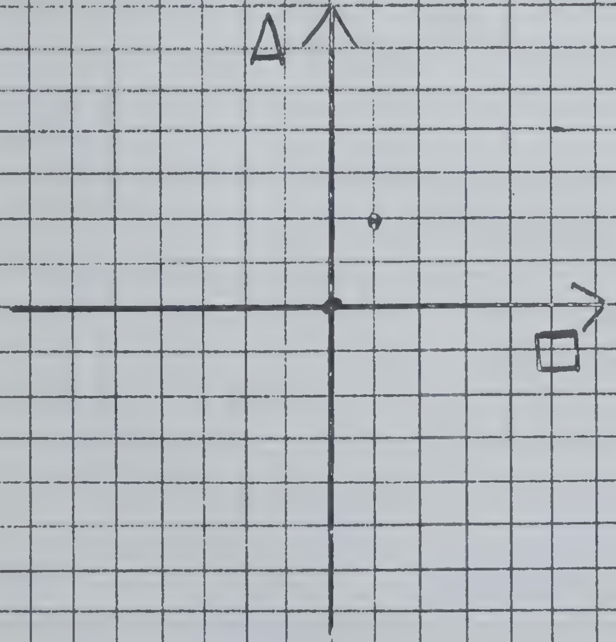

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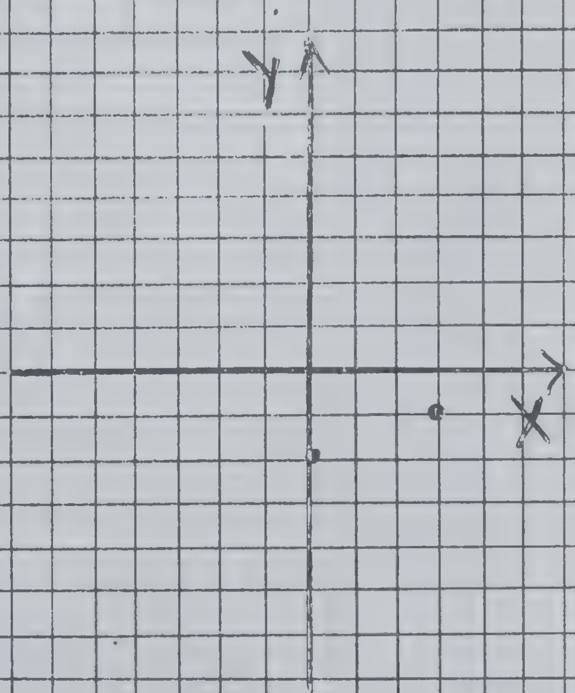

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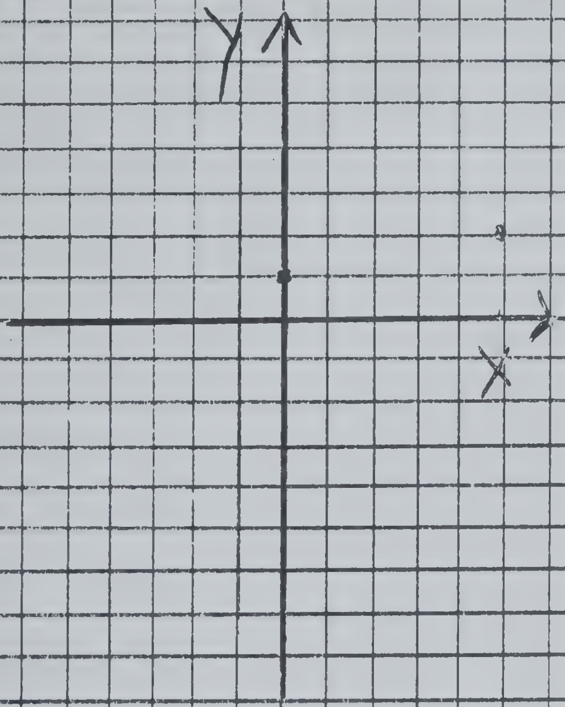
21)



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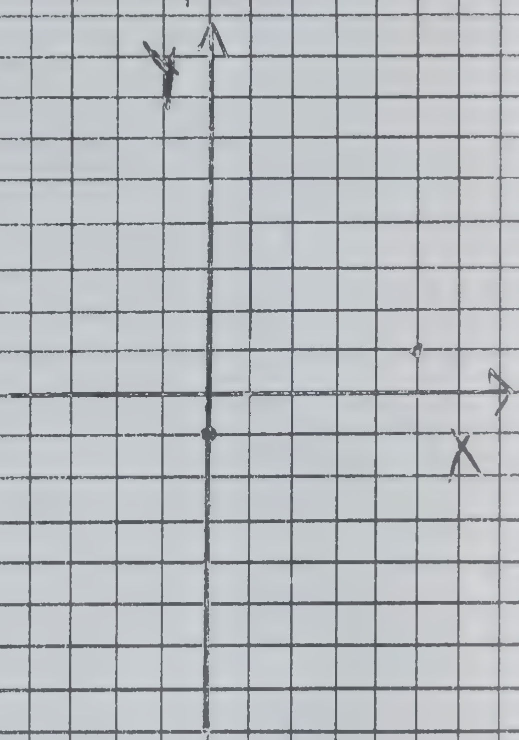

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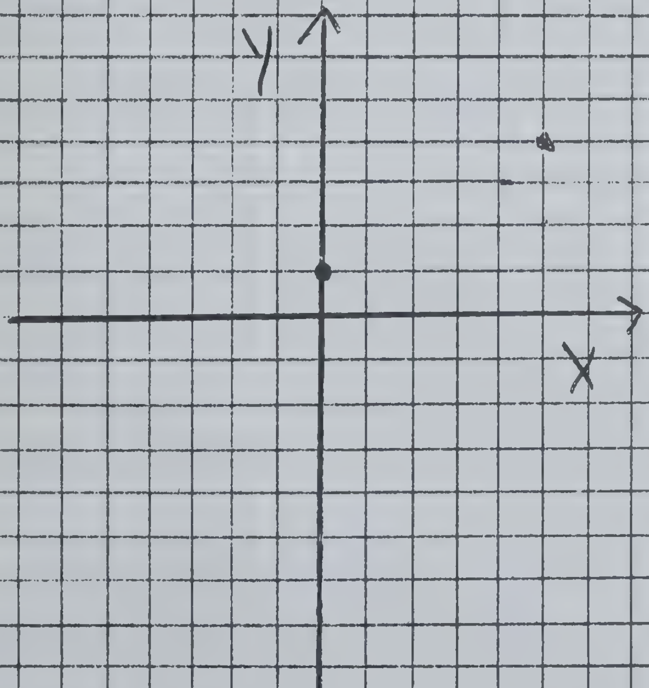

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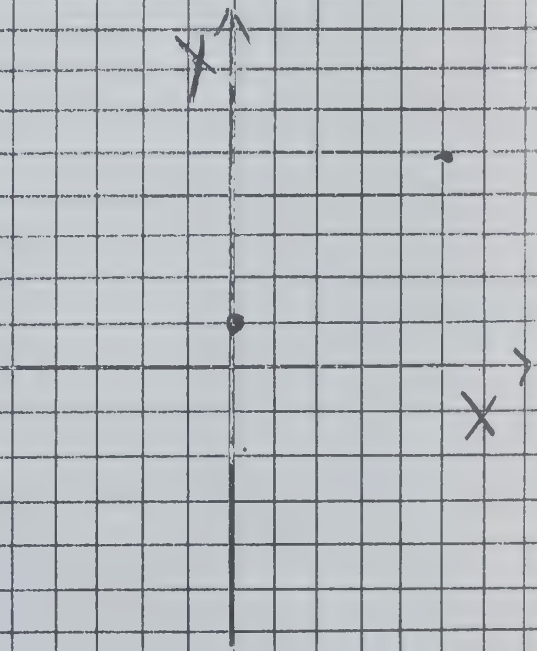

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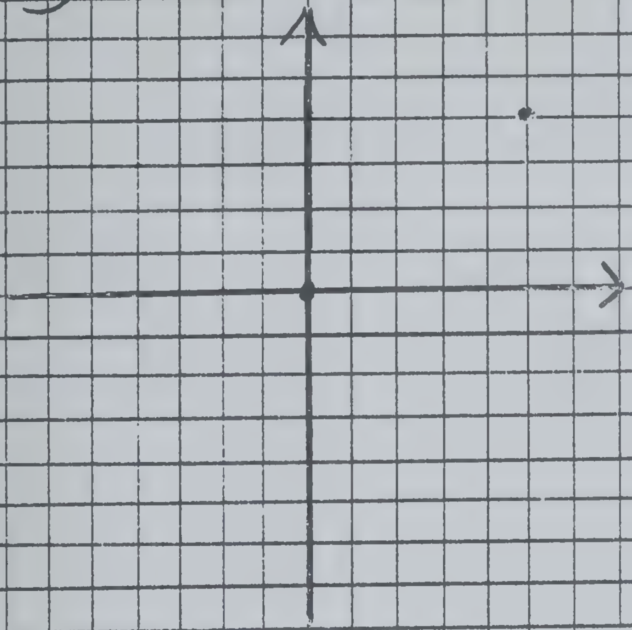
25)



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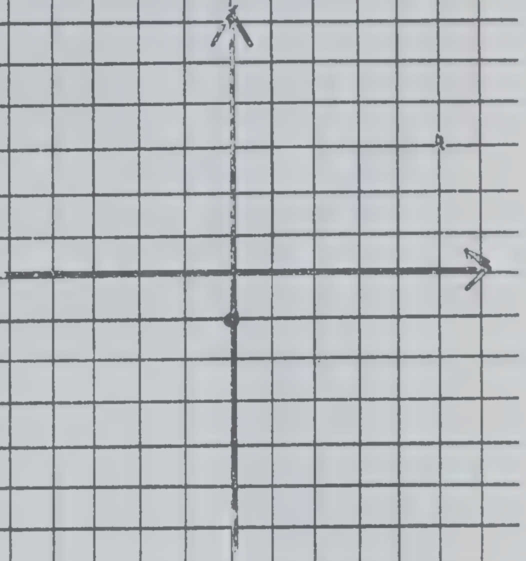

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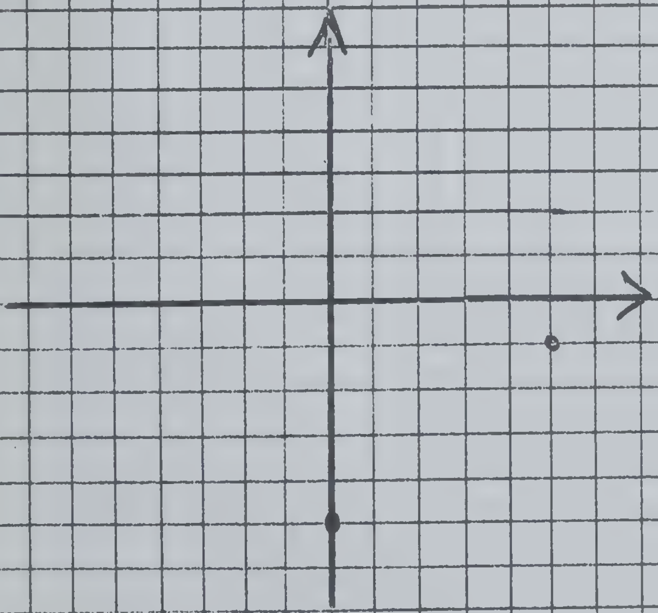

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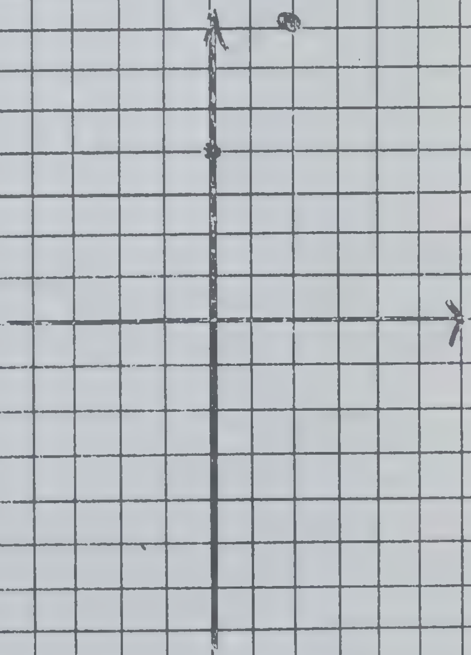

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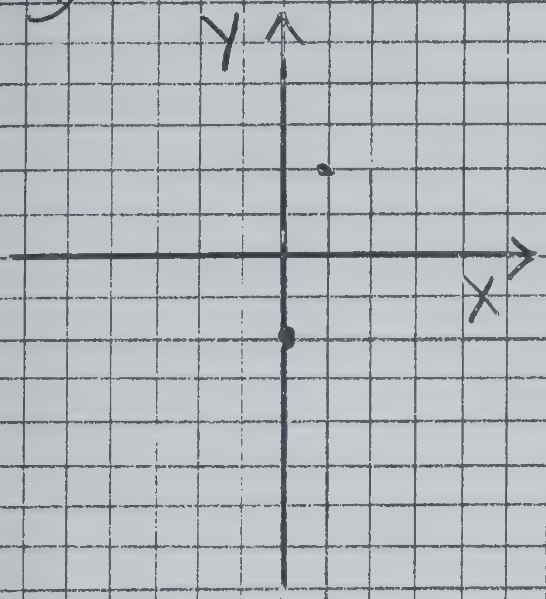
29)



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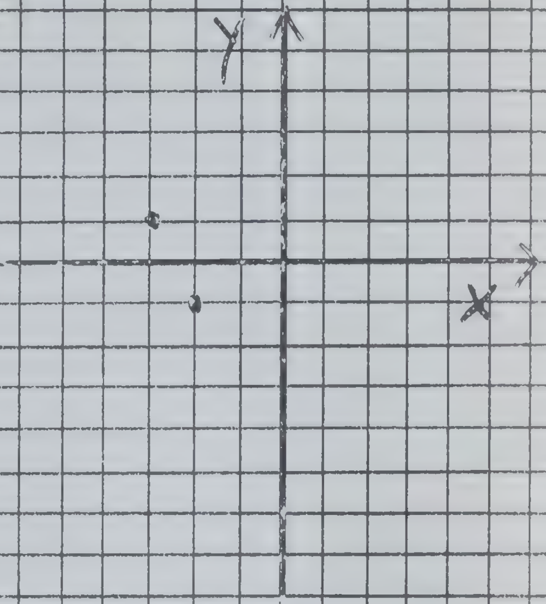

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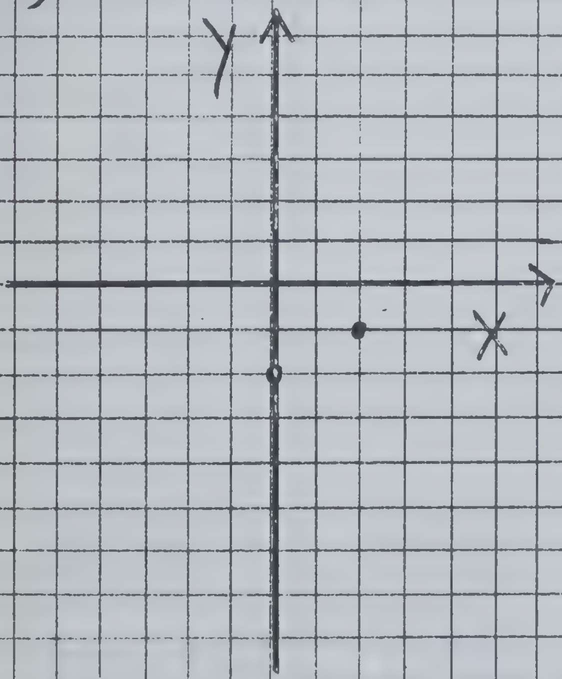

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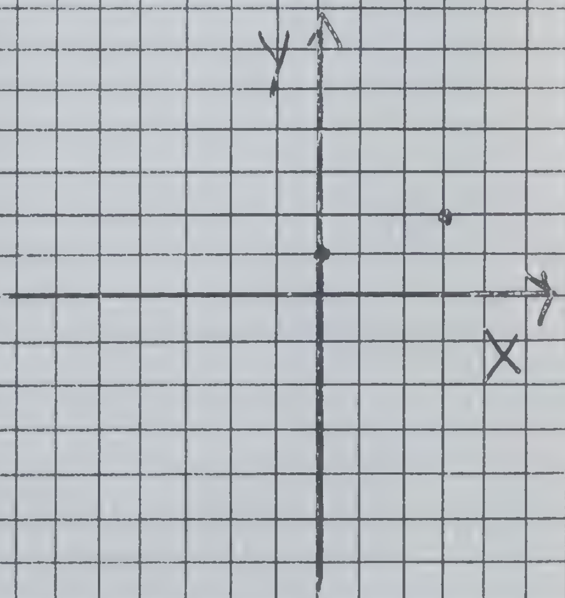

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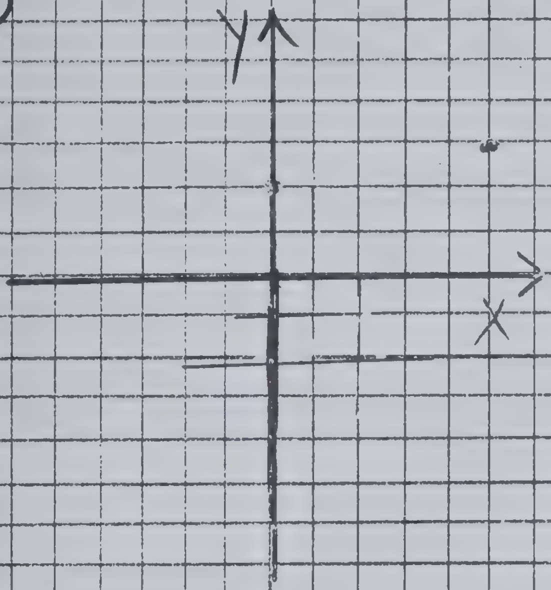
33)



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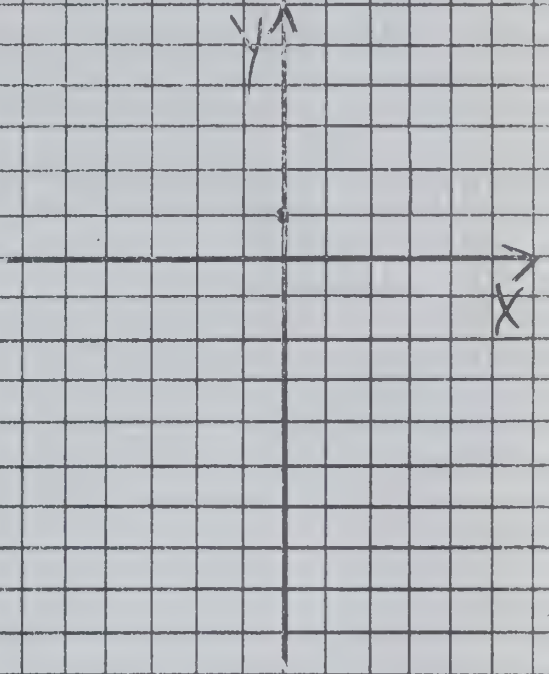

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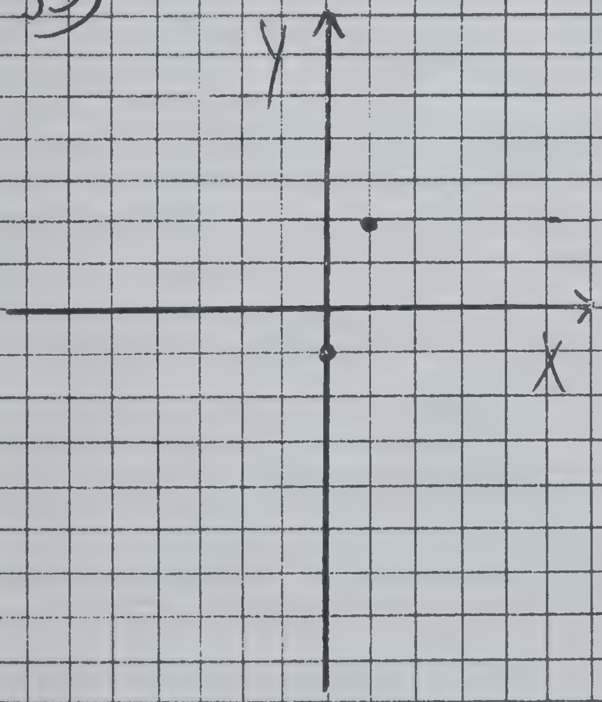

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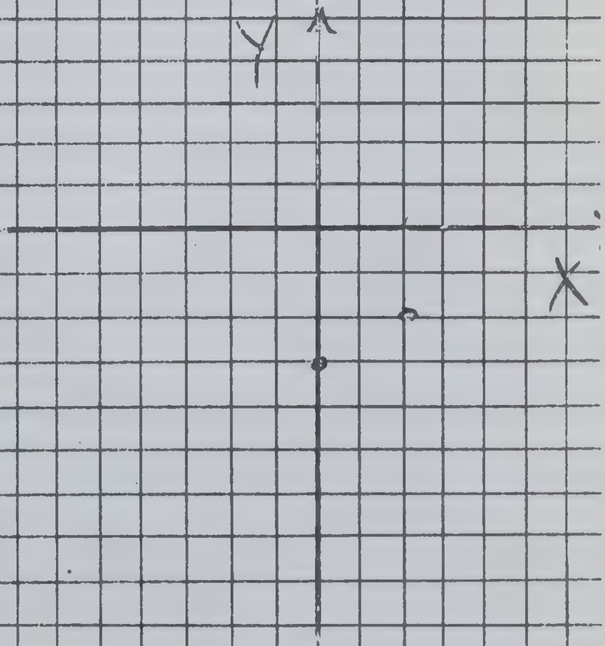

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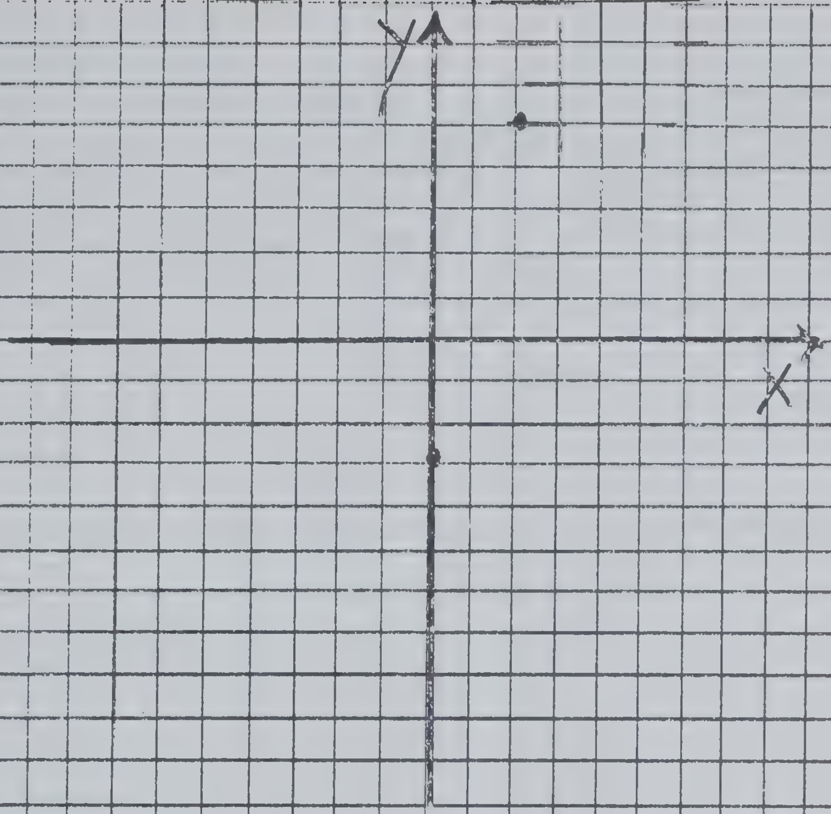
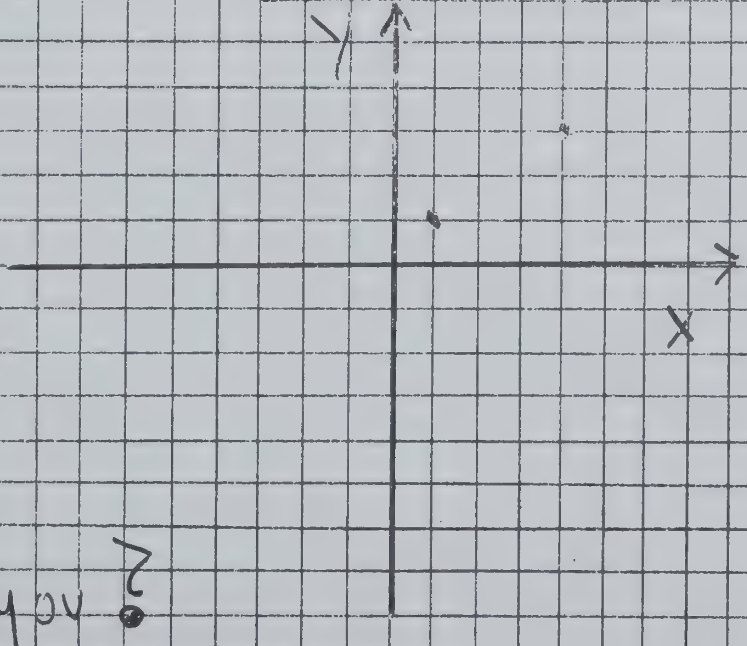


32) Jerry made up this graph. He says nobody in the class can write its equation:



Can you?

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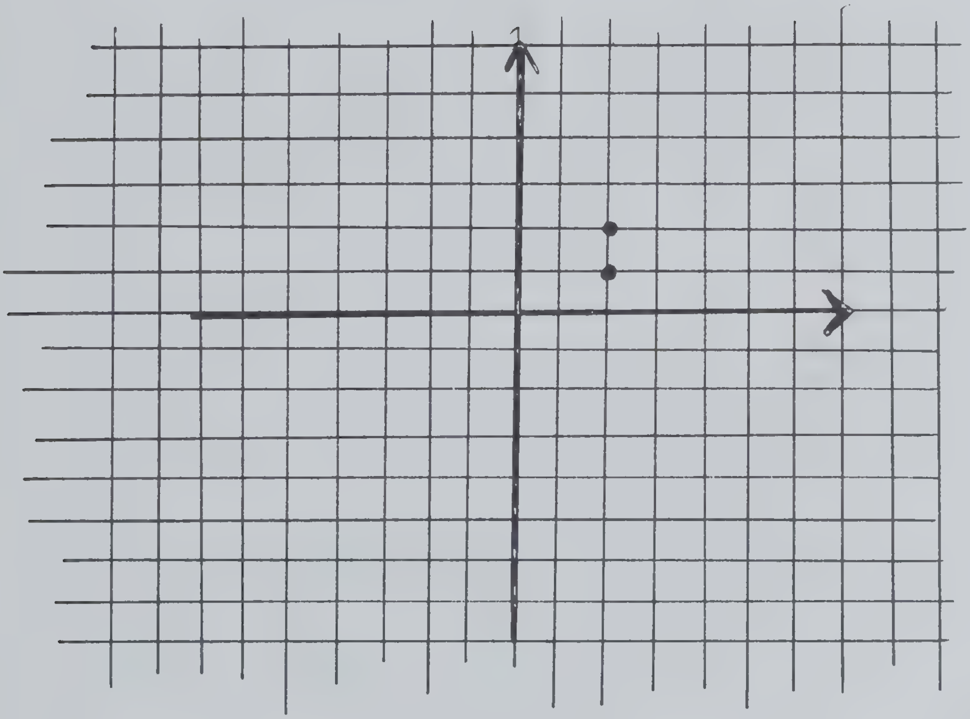
39) Al says that positively nobody can find the equation for this straight line graph. Can you?



37 Finding Truth Sets

Which statements are True? False? Open?

1) $\frac{1}{2} + \frac{1}{2} = 1$

2) $3 + 7 = 11$

3) $14 + 6 = 20$

4) $8 + \square = 12$

5) $\frac{1}{2} + \frac{1}{4} = 1\frac{1}{2}$

6) $(\square \times \square) = 121$

7) $14 + 26 = 30$

8) $19 + 2 = 21$

9) $3 < 5$

10) $7 < 15$

11) $3 < \square < 11$

12) $1 < \square < 5$

13) $8 < 3$

14) $53 + 37 = 80$

15) $8 \times 8 = 16$

16) ${}^{+}2 \times {}^{-}3 = {}^{-}6$

17) ${}^{-}2 \times {}^{-}2 = {}^{+}4$

18) ${}^{+}5 < {}^{+}7$

19) $0 < {}^{+}3$

20) ${}^{-}2 < {}^{+}1$

21) $31 + 19 = 60$

22) $4 \times 12 = 64$

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Can you find the truth sets for these open sentences?

23) ______________ sit in the front row.

24) Today is ______________.

25) $3 + \square = 8$

26) $11 + (2 \times \square) = 16$

27) $5 + (2 \times \square) = 207$

28) $2 + (2 \times \square) = 208$

29) $8 + (2 \times \square) = 13$

30) $3 \times \square = 16$

31) $5 + (2 \times \square) = 206$

32) $3 + (11 \times \square) = 58$

33) $2 + (11 \times \square) = 36$

34) $4 + (7 \times \square) = 19$

35) $3+(4\times\square)=10$

36) $(\square\times\square)-(5\times\square)+6=0$

37) $(\square\times\square)-7\times\square)+12=0$

38) $(\square\times\square-(53\times\square)+150=0$

39) $(\square\times\square)-(13\times\square)+22=0$

40) $(\square\times\square)-(15\times\square)+50=0$

41) $(\square\times\square)-(102\times\square)+200=0$

42) $(\square\times\square)-(15\times\square)+26=0$

43) $(\square\times\square)-(20\times\square)+99=0$

44) $(\square\times\square)-(14\times\square)+48=0$

45) $(\square\times\square)-(26\times\square)+48=0$

46) $(\square\times\square)-(15\times\square)+36=0$

47) $(\square\times\square)-(60\times\square)+500=0$

48) $(\square\times\square)-(3\times\square)+2=0$

49) $(\square \times \square) - (5 \times \square) + 0 = 0$

50) $(\square \times \square) - (20 \times \square) + 96 = 0$

51) $(\square \times \square) - (35 \times \square) + 300 = 0$

52) $(\square \times \square) - (77 \times \square) + 150 = 0$

53) $(\square \times \square) - (8 \times \square) + 15 = 0$

54) $(\square \times \square) - (103 \times \square) + 300 = 0$

55) $(\square \times \square) - ({}^{+}5 \times \square) + {}^{-}24 = 0$

56) $(\square \times \square) - ({}^{+}3 \times \square) + {}^{-}10 = 0$

57) $(\square \times \square) - ({}^{+}5 \times \square) + {}^{-}14 = 0$

58) $(\square \times \square) - ({}^{-}2 \times \square) + {}^{-}3 = 0$

59) $(\square \times \square) - ({}^{-}8 \times \square) + {}^{+}15 = 0$

60) $(\square \times \square) - ({}^{-}15 \times \square) + {}^{+}50 = 0$

61) $(\square \times \square) - ({}^{-}10 \times \square) + {}^{+}16 = 0$

62) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{-}50 = 0$

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63) $(\square \times \square) - ({}^{+}10 \times \square) + {}^{+}21 = 0$

64) $5 + (2 \times \square) = \square + 11$

65) $8 + (2 \times \square) = \square + 19$

66) $5 + (2 \times \square) = \square + 25$

67) $5 + (2 \times \square) = \square + 7$

68) $5 + (2 \times \square) = \square + 20$

69) $5 + (3 \times \square) = \square + 17$

70) $5 + (3 \times \square) = \square + 15$

71) $5 + (3 \times \square) = \square + 13$

72) $5 + (3 \times \square) = \square + 7$

73) $5 + (3 \times \square) = \square + 205$

74) $3 + (3 \times \square) = \square + 83$

75) $8 + (3 \times \square) = \square + 78$

76) $3 + (2 \times \square) = \square + 11$

77) $10 + (2 \times \square) = \square + 12$

78) $3 + (5 \times \square) = \square + 43$

79) $21 + (3 \times \square) = \square + 31$

80) $8 + \square = 20$

81) $3 + (2 \times \square) = 23$

82) $1 + (5 \times \square) = \square + 9$

83) $1 + (5 \times \square) = (3 \times \square) + 31$

84) $1 + (5 \times \square) = 31$

85) $7 = (2 \times \square) + 1$

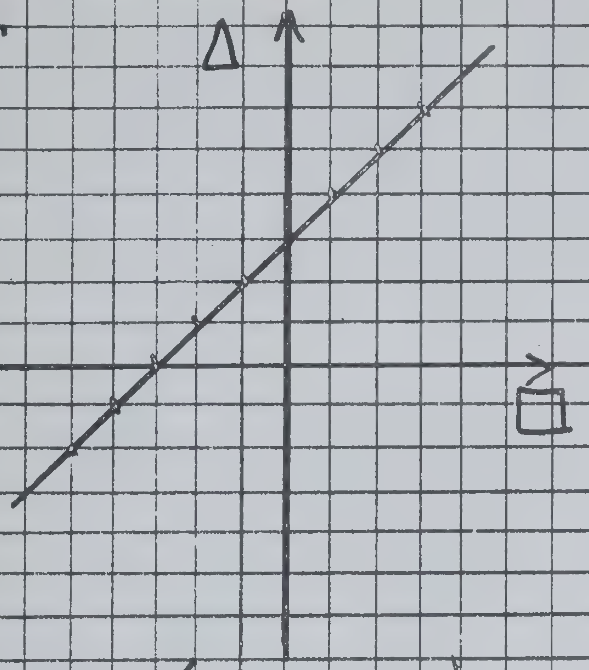
86) $3 + (2 \times \square) = 5 + (2 \times \square)$

87) $1 + (5 \times \square) = (2 \times \square) + 61$

88) $1 + (10 \times \square) = (5 \times \square) + 36$

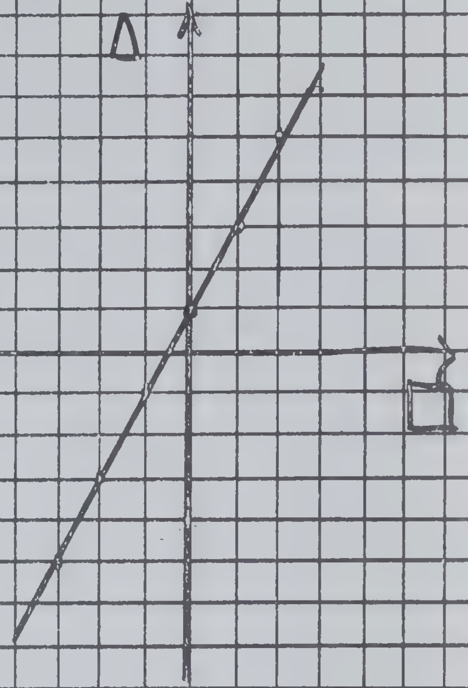
Can you find the equations for these graphs?

89.



$\triangle = (\underline{\hspace{3em}} \times \square) + \underline{\hspace{3em}}$

90.



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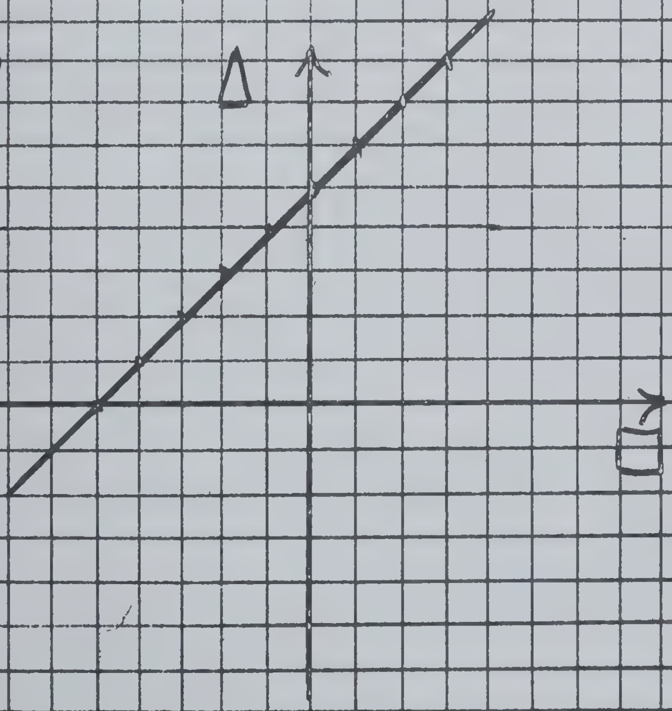

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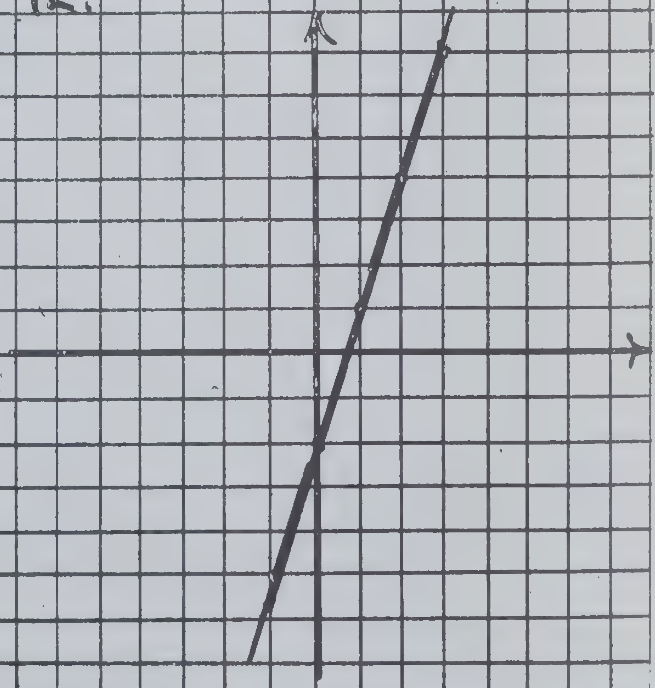

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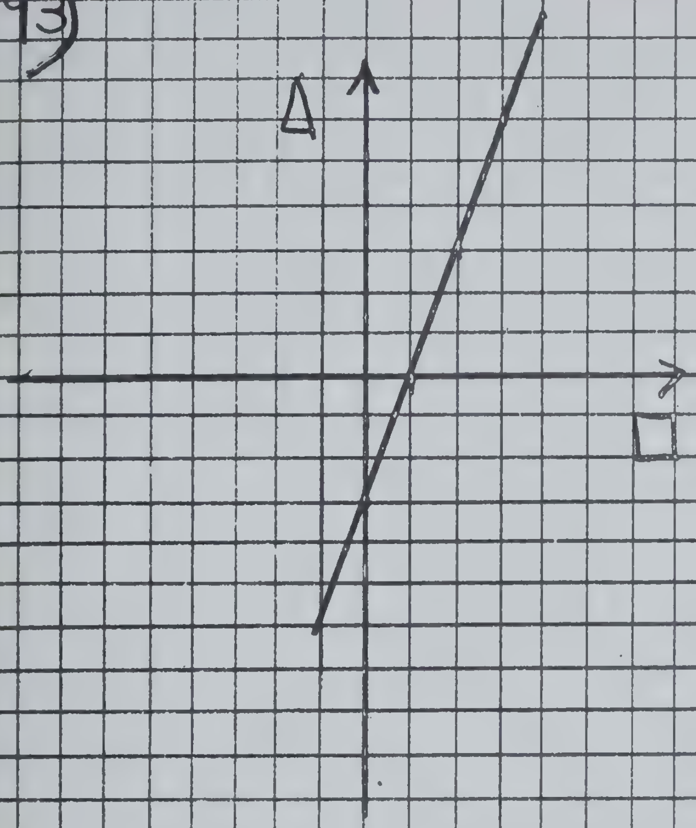
93)



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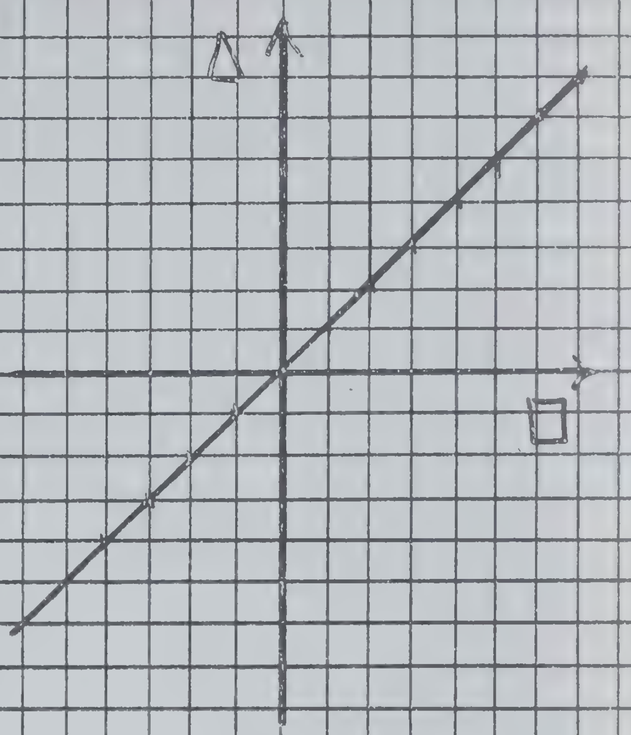

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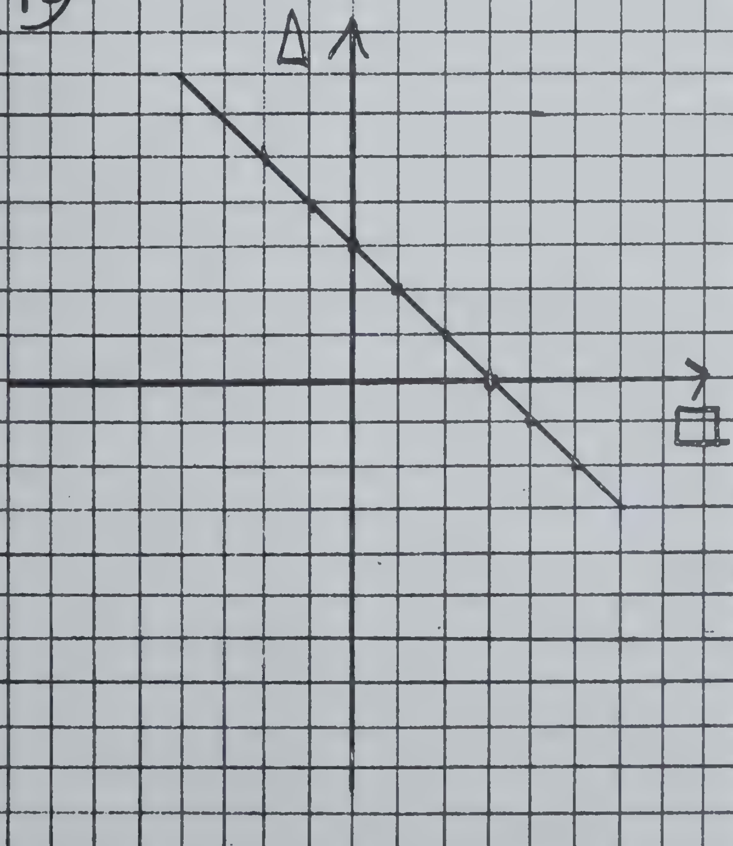

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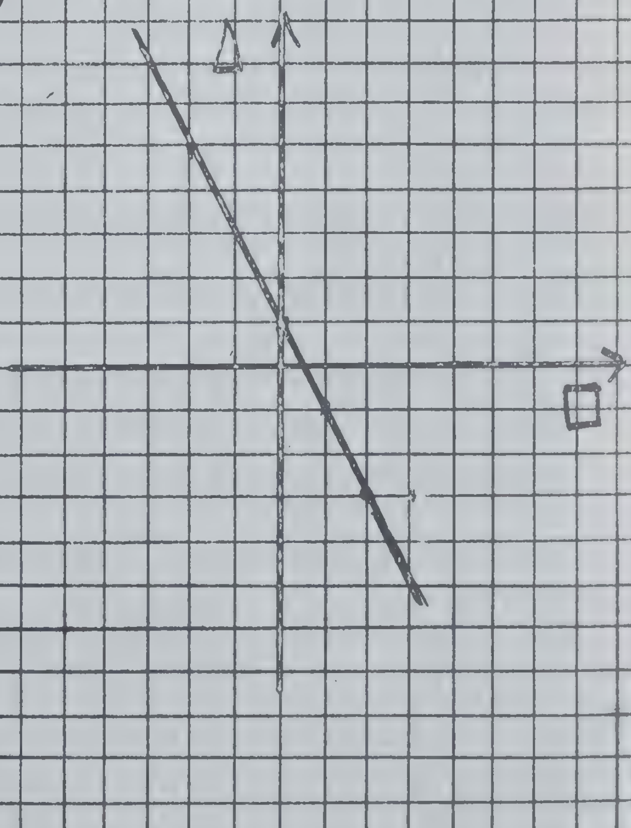

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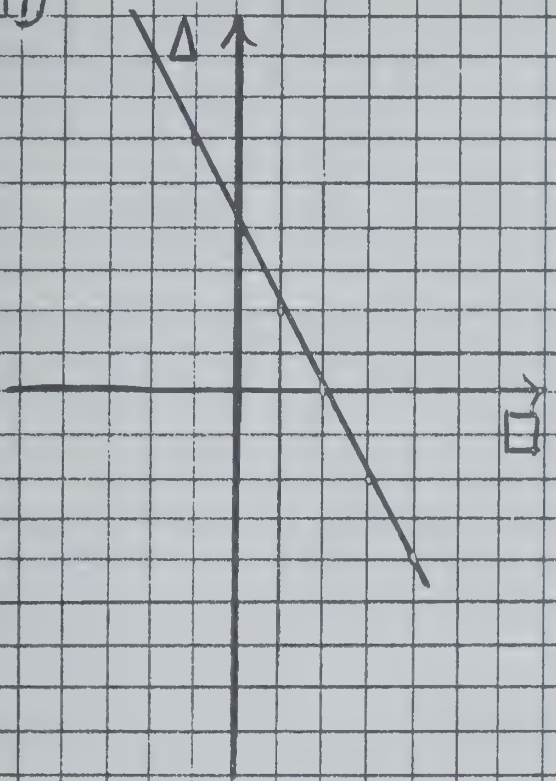

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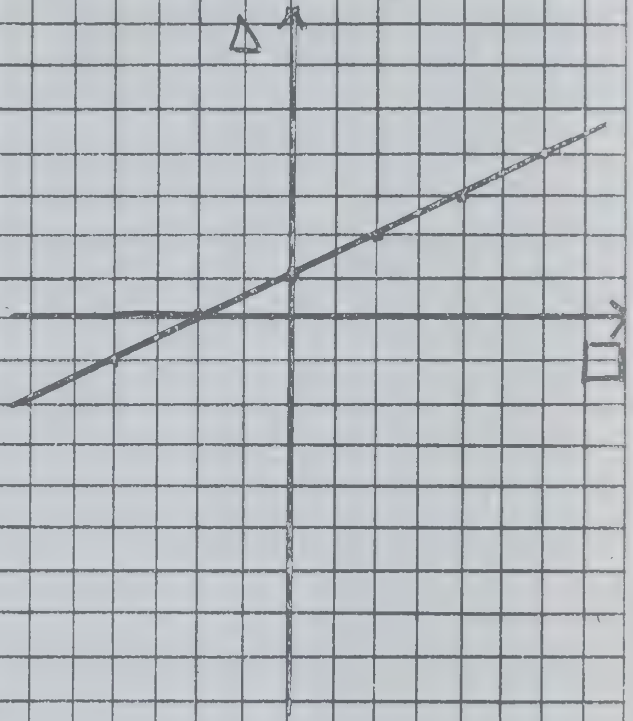

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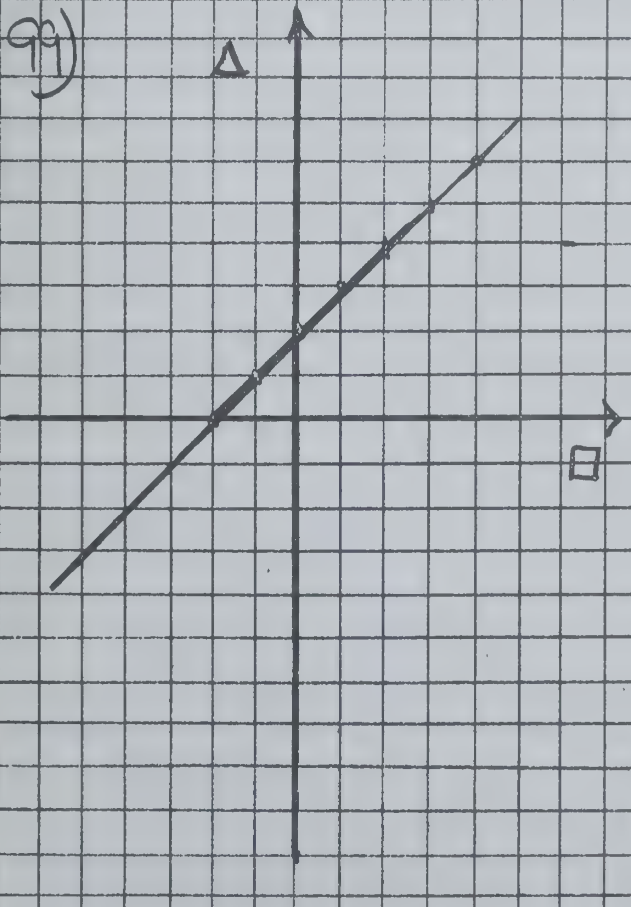

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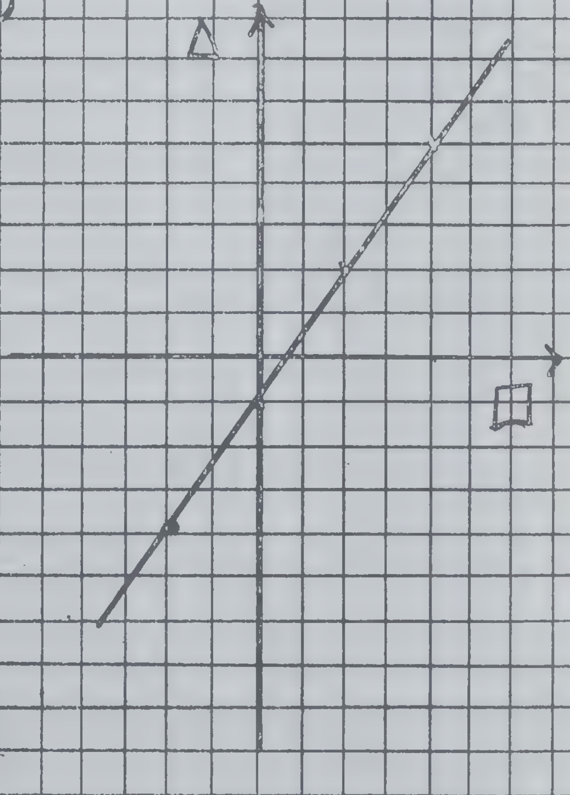

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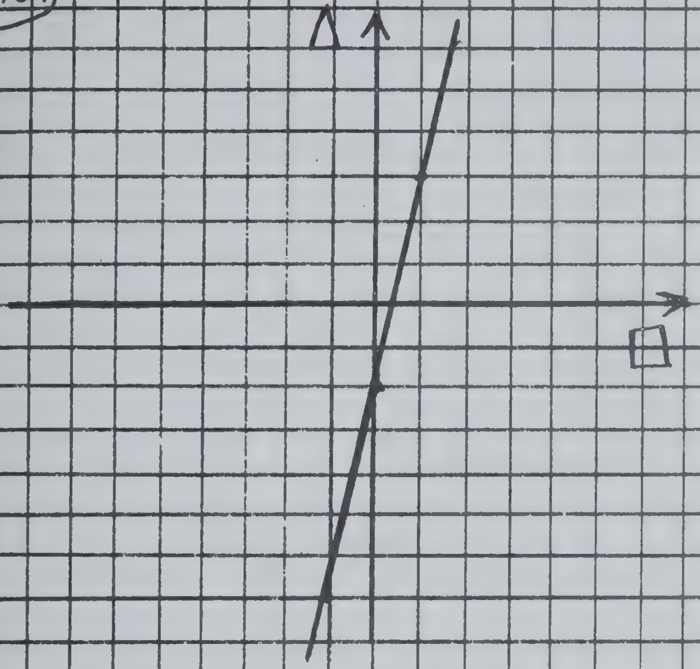
101)



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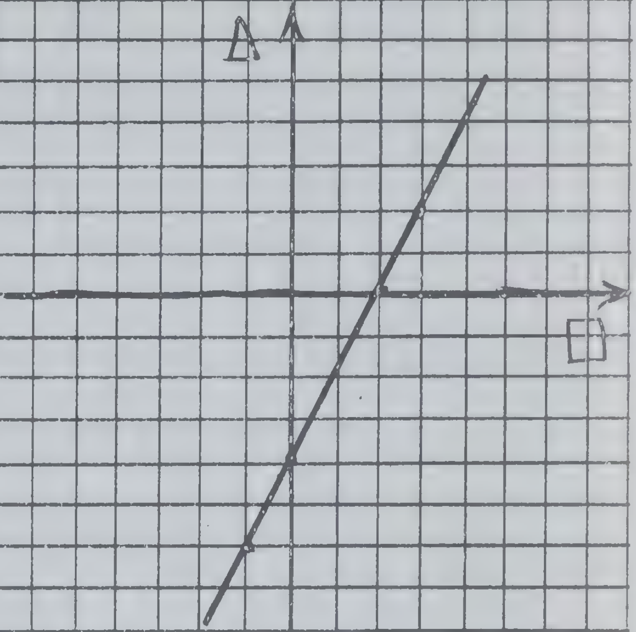

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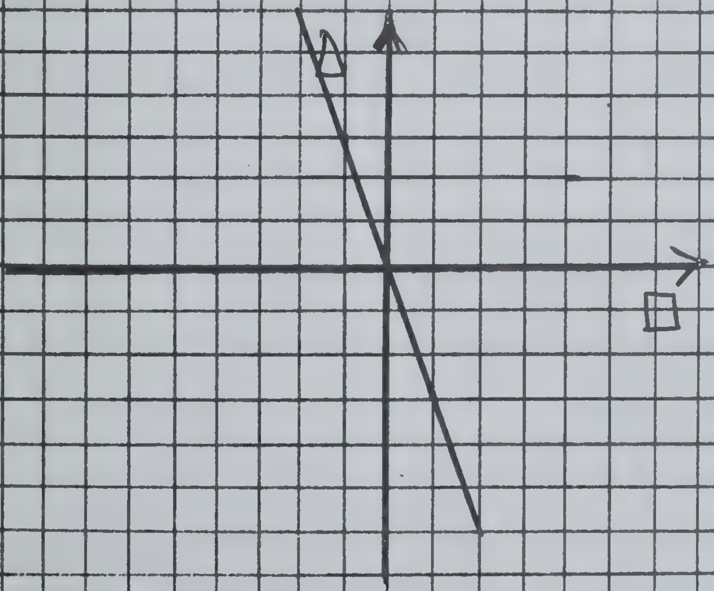

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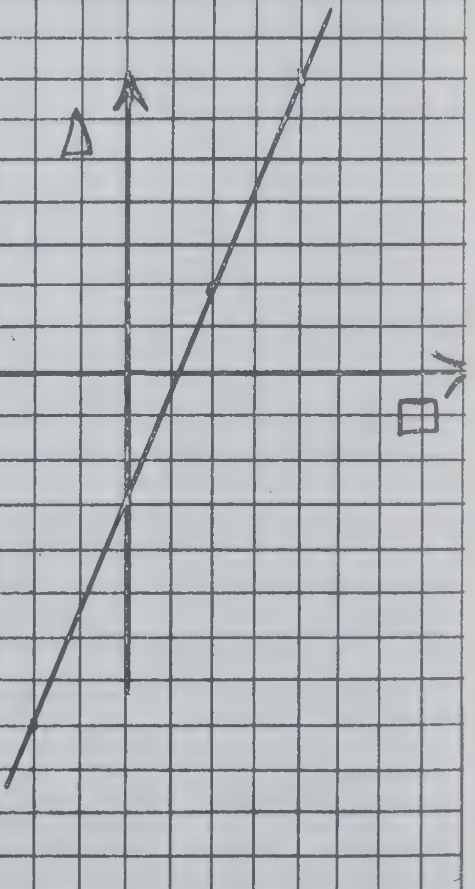

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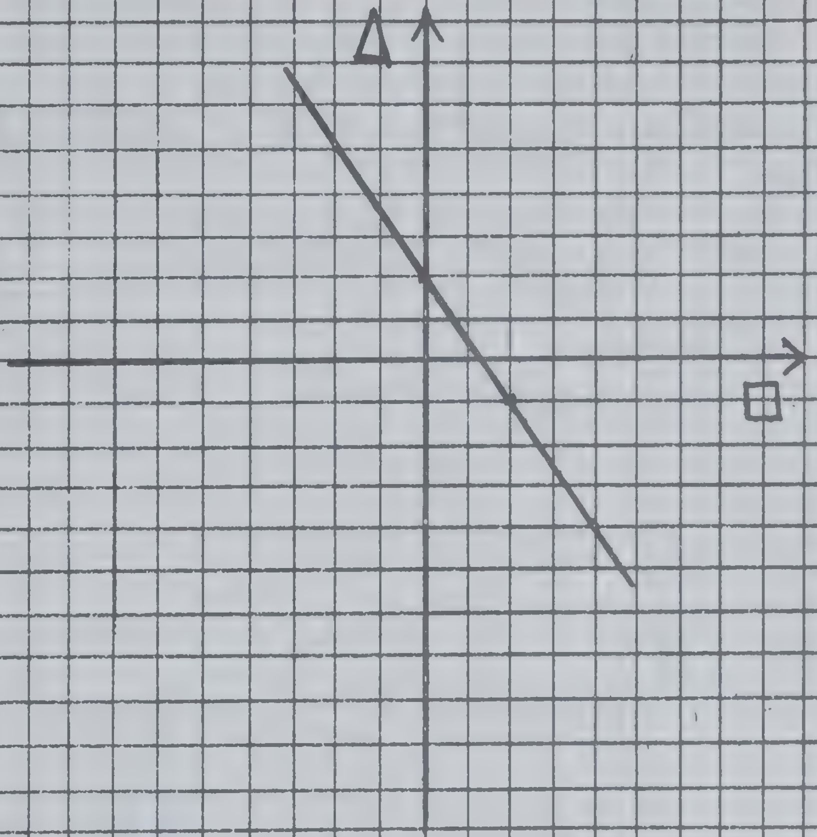
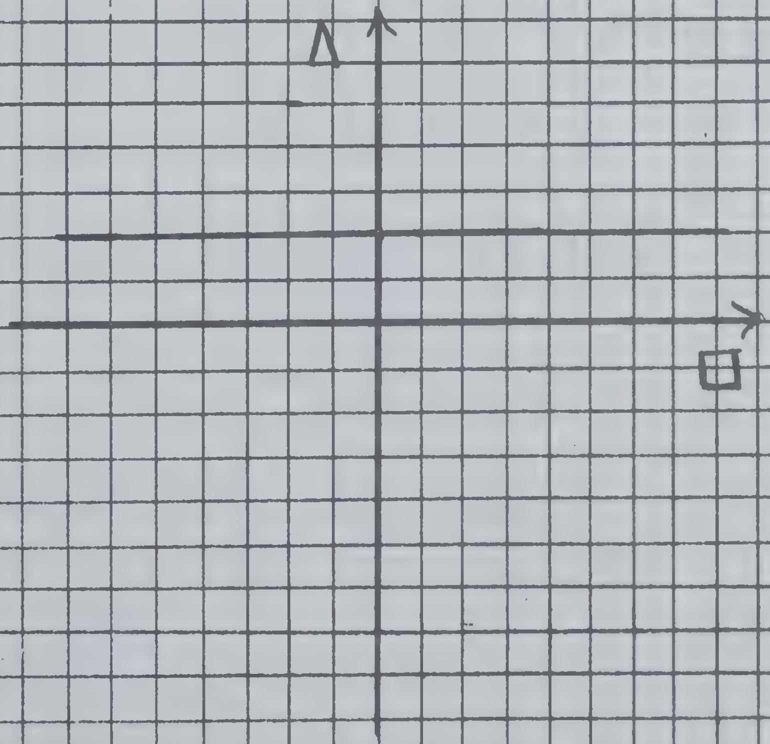
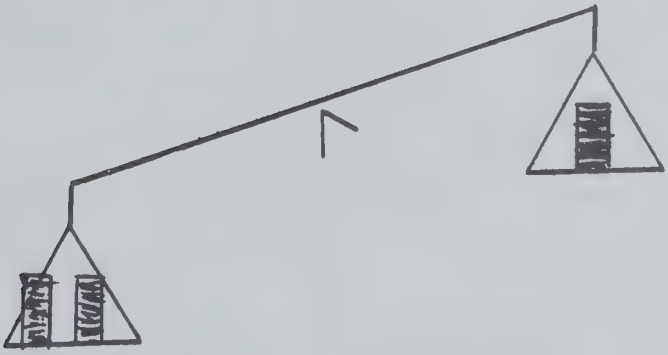


106)



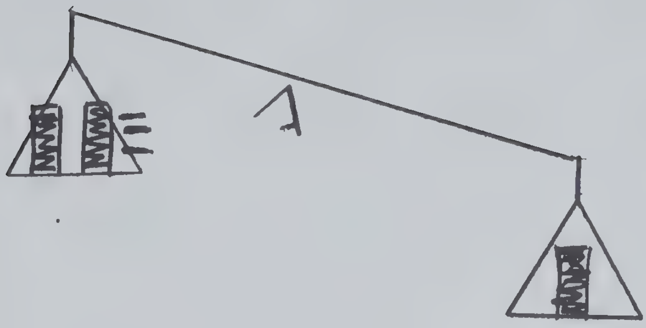
38. Balance Pictures

1) Debbie has three rolls of dimes, each wrapped in paper. They are on a balance like this:

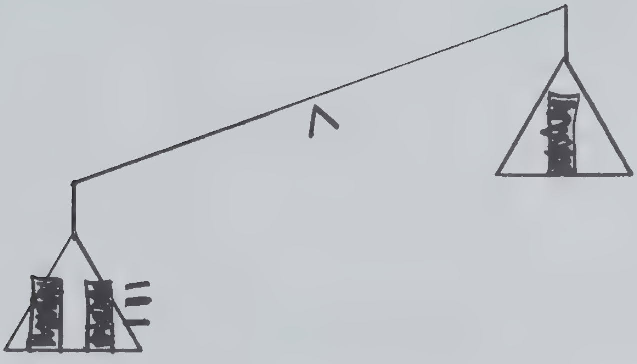


Can you tell how many dimes there are in each roll?

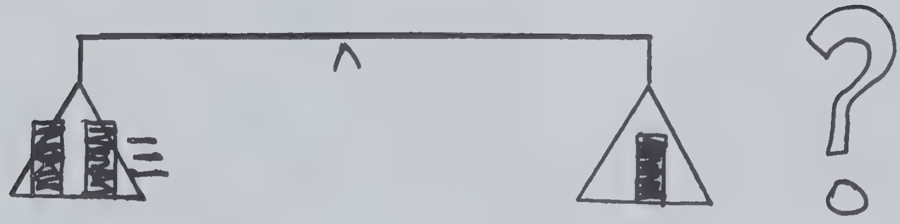
2) When Debbie added 3 loose dimes to the left balance, what did the balance do? Did it look like this?



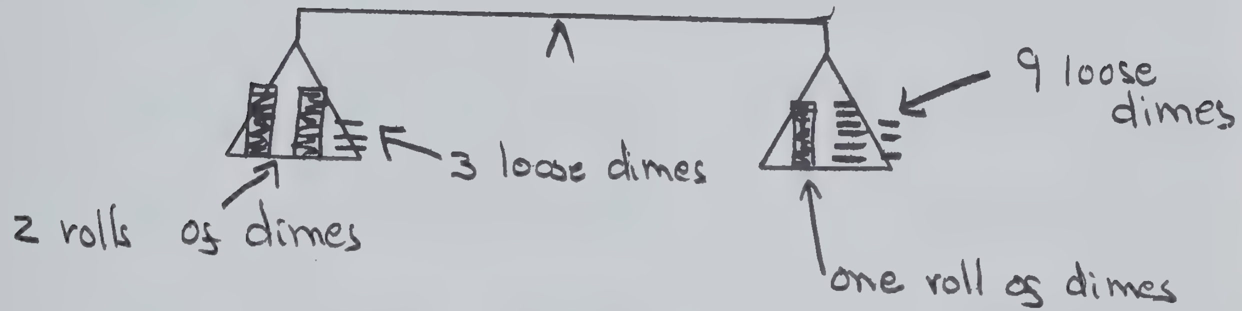
or like this



or like this



3) Now Debbie added 9 loose dimes to the right hand side, and the balance looked like this:



Can you tell how many dimes there are in each roll?

4) Sandy says this problem is too complicated. She says that the balance balances correctly now, and if you take one roll of dimes off each side, it will still balance

correctly. Is Sandy right?

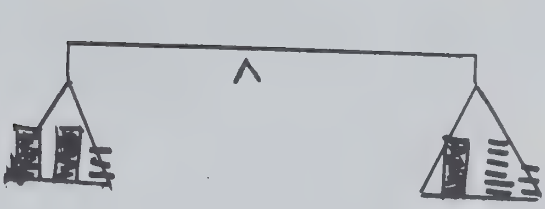
5) What does the balance look like after Sandy removes one roll of dimes from each side?

6) Richard says you can make the problem even simpler, by taking 3 loose dimes off each side. What does the balance look like after Richard removes 3 loose dimes from each side?

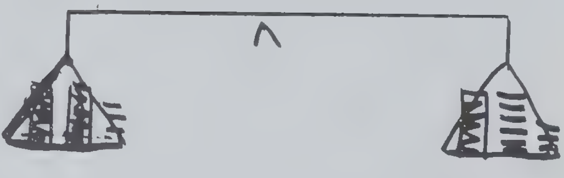
7) Can you tell how many dimes there are in each roll?

8) Alex says that each balance picture matches an equation:

$(2 \times \square) + 3 = (1 \times \square) + 9$



$(2 \times \square) + 1 = (1 \times \square) + 7$



What did Alex take off, to get from the first picture to the second? Will the scales still balance?

9) Suppose you now remove one roll of climes from each side. Will the scales still balance?

10) What do the scales look like now?

11) Can you write an equation that

matches the way the scales look now?

For each equation, can you make a balance picture? Can you take something off to make the problem easier?

12) $(5 \times \square) + 3 = (4 \times \square) + 14$

13) $(5 \times \square) + 6 = (3 \times \square) + 16$

14) $(5 \times \square) + 5 = (3 \times \square) + 35$

15) $(5 \times \square) + 4 = (3 \times \square) + 24$

16) What name do we use to describe

two equations that have the same truth set?

17) What do we mean by a "transform operation"?

Can you make a balance picture for each equation? Can you find the truth set?

18) $(4 \times \square) + 5 = (3 \times \square) + 12$

19) $(6 \times \square) + 3 = (2 \times \square) + 11$

20) $(7 \times \square) + 4 = (3 \times \square) + 36$

21) $(7 \times \square) + 5 = (4 \times \square) + 23$

39) Open Sentences

Can you find the truth set for each open sentence?

1) $5+(2\times\square)=19$
2) $3+(2\times\square)=204$
3) $7+(2\times\square)=112$
4) $2+(2\times\square)=5$
5) $^{+}7+\square={}^{+}4$
6) $^{+}12+\square={}^{+}6$
7) $3+(4\times\square)=12$
8) $2+(3\times\square)=18$
9) $2+(5\times\square)=38$
10) $5+(3\times\square)=10$
11) $(\square\times\square)-(5\times\square)+6=0$

12) $(\square \times \square) - (7 \times \square) + 10 = 0$

13) $(\square \times \square) - (28 \times \square) + 75 = 0$

14) $(\square \times \square) - (107 \times \square) + 700 = 0$

15) $(\square \times \square) - (6 \times \square) + 8 = 0$

16) $(\square \times \square) - (60 \times \square) + 800 = 0$

17) $(\square \times \square) - ({}^{+}5 \times \square) + {}^{-}14 = 0$

18) $(\square \times \square) - ({}^{+}9 \times \square) + {}^{-}22 = 0$

19) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{+}6 = 0$

20) $(\square \times \square) - ({}^{-}8 \times \square) + {}^{+}7 = 0$

21) $(\square \times \square) - (3 \times \square) + 2 = 0$

22) $(5 \times \square) + 3 = (4 \times \square) + 15$

23) $(5 \times \square) + 6 = (4 \times \square) + 9$

24) $(5 \times \square) + 2 = (4 \times \square) + 28$

25) $(5 \times \square) + 8 = (3 \times \square) + 18$

26) $(5 \times \square) + 12 = (3 \times \square) + 52$

27) $(7 \times \square) + 9 = (4 \times \square) + 15$

28) $(\square \times \square) - (8 \times \square) + 21 = 6$

29) $(\square \times \square) - (17 \times \square) + 86 = 20$

30) $(\square \times \square) - (25 \times \square) + 225 = 75$

31) $(\square \times \square) - (8 \times \square) + 214 = 202$

32) $(\square \times \square) - (13 \times \square) + 93 = 53$

33) $(2 \times \square) + 5 = 9 + 2$

Can you write the equations for these graphs?

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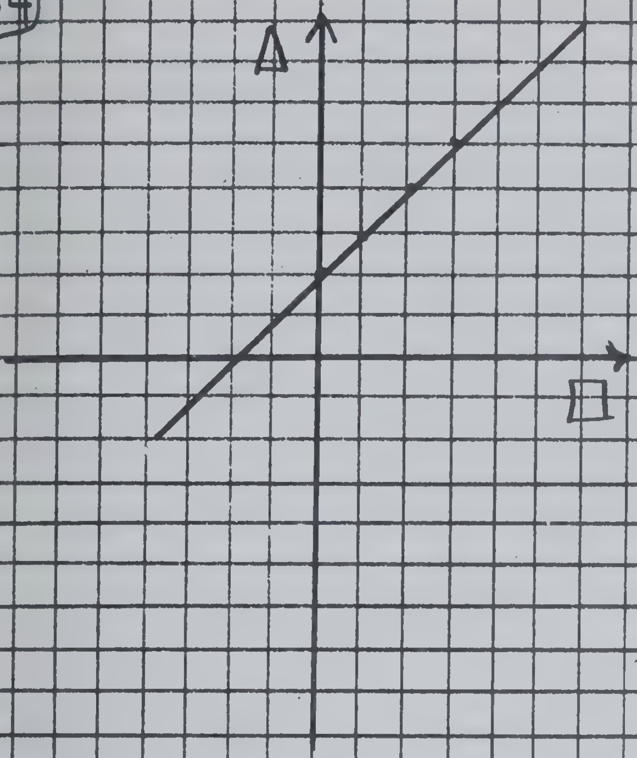
34)



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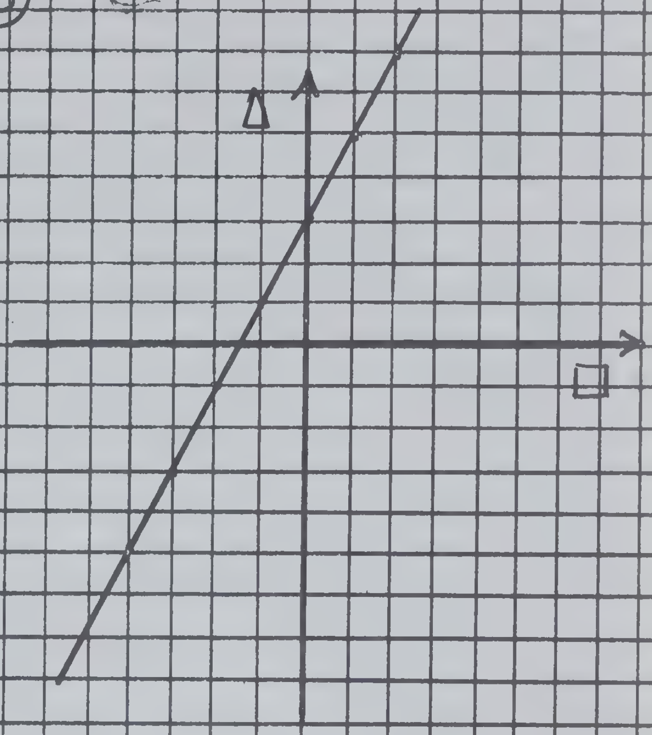
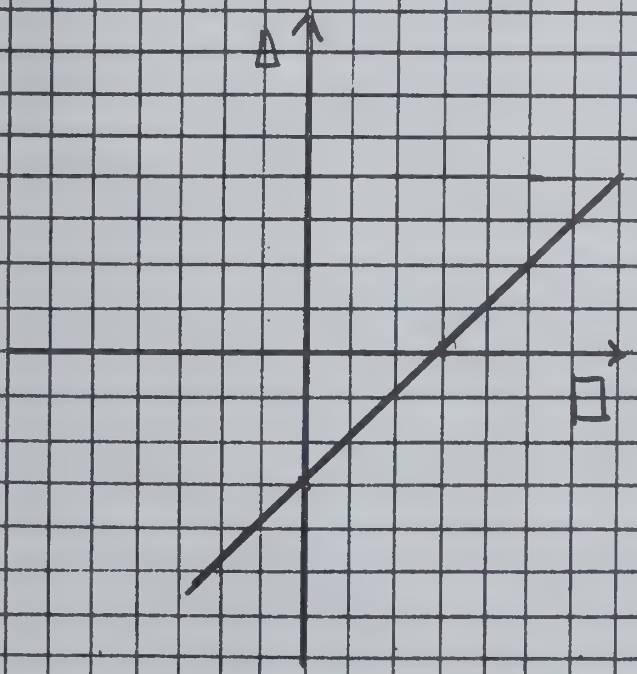
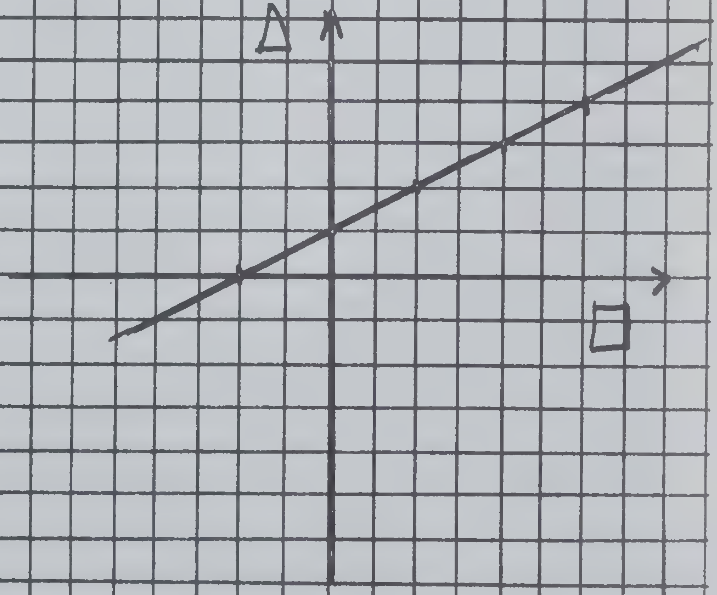


36)



37)



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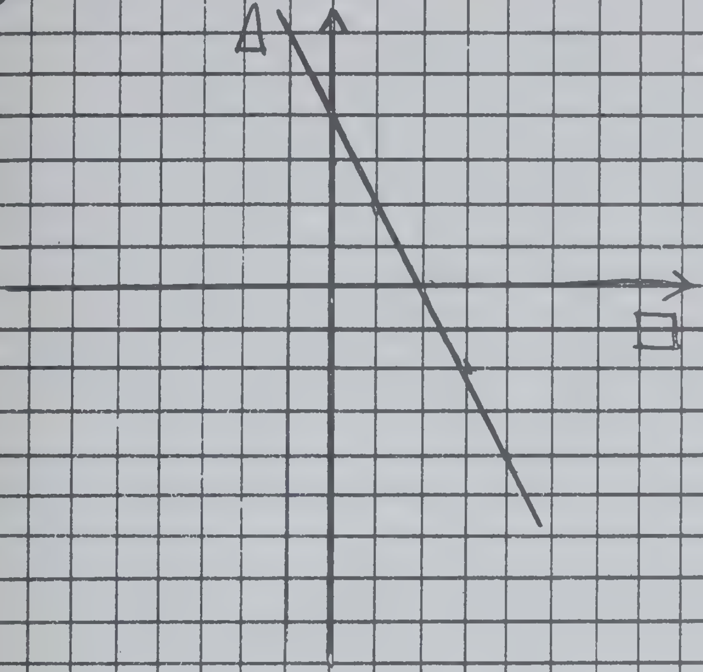
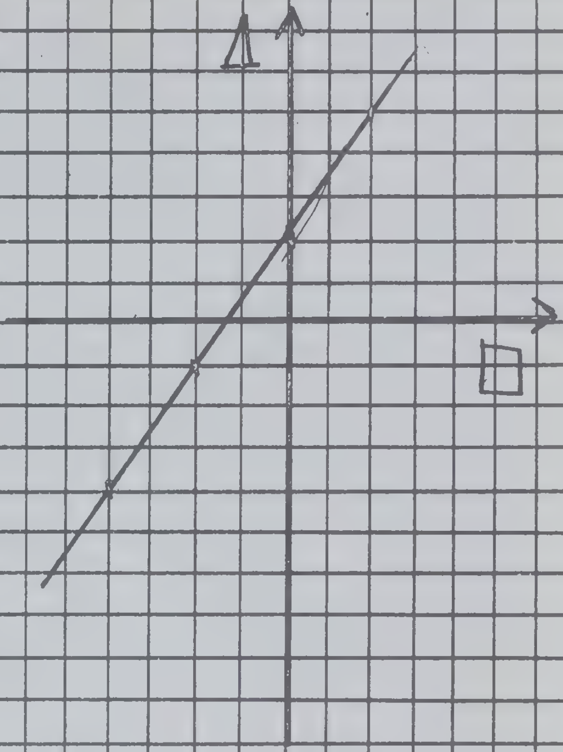
38)



39)



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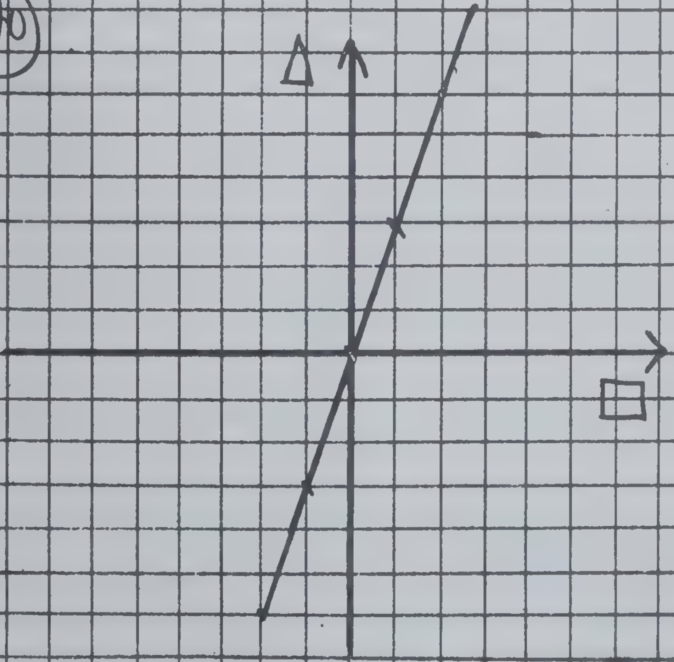

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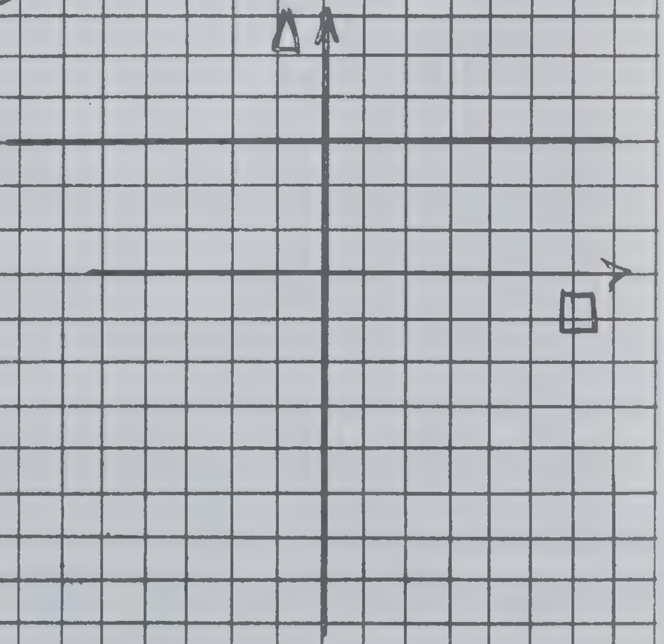
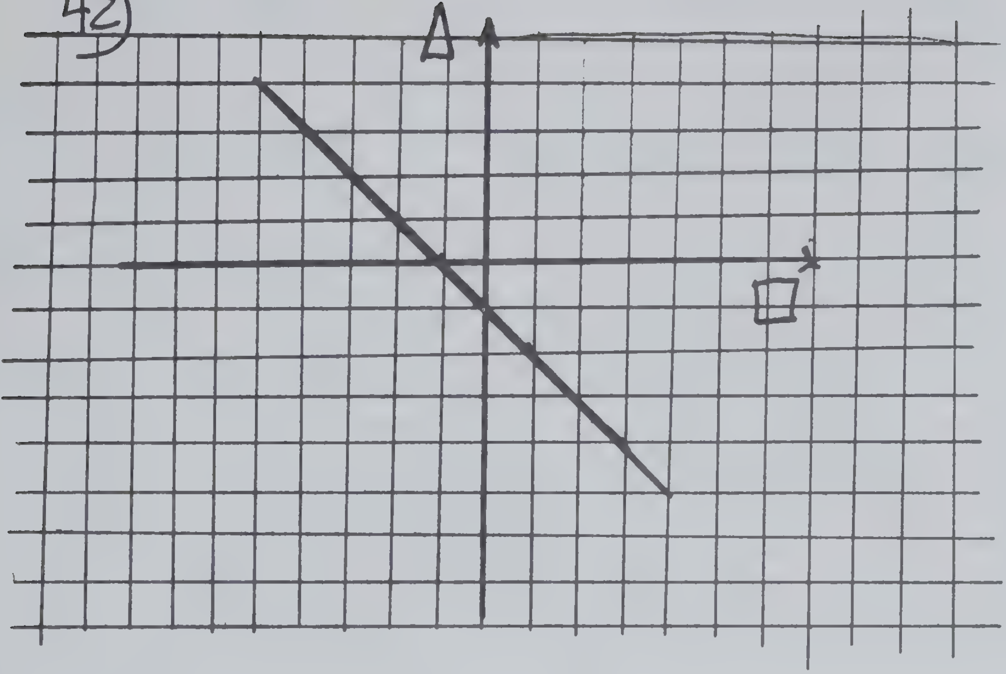
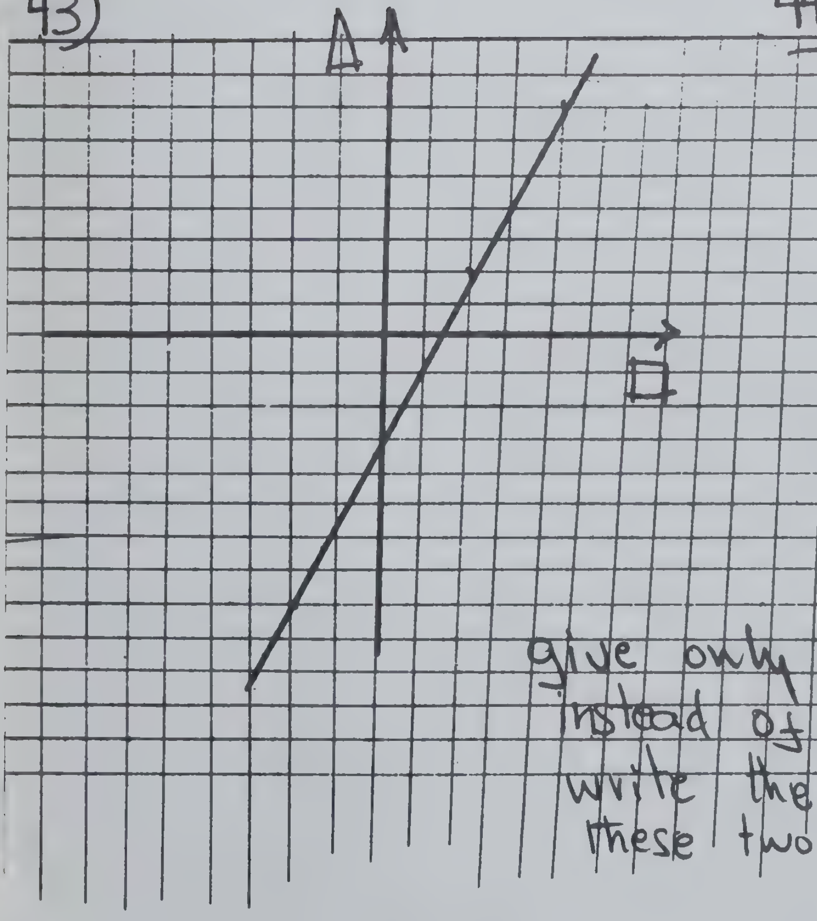


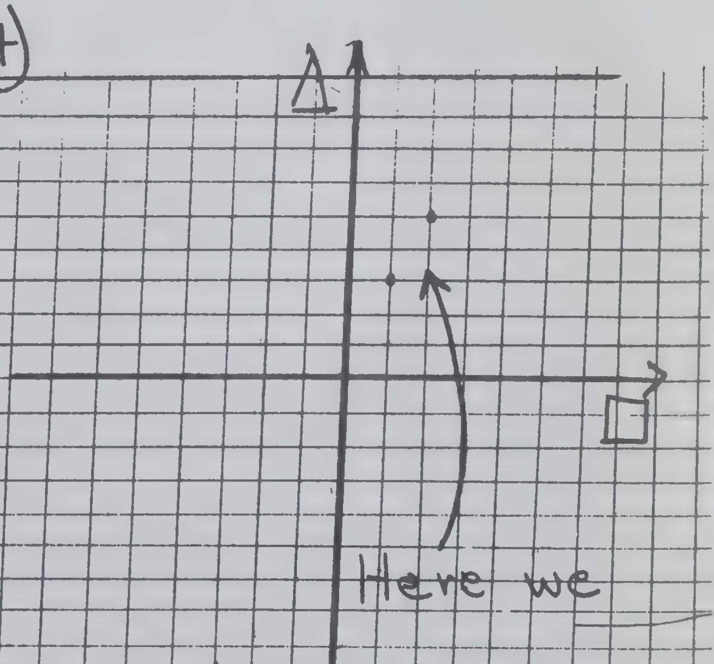
42)



43)



44)



Here we give only 2 points on the line, instead of the whole line. Can you write the equation of the line through these two points?

Can you make a balance picture for each equation? Can you find the truth set?

45) $(5 \times \square) + 3 = (4 \times \square) + 20$

46) $(7 \times \square) + 5 = (4 \times \square) + 50$

47) $(8 \times \square) + 9 = (2 \times \square) + 27$

48) Lex and Debbie made these balance pictures and equations, in order to solve the equation

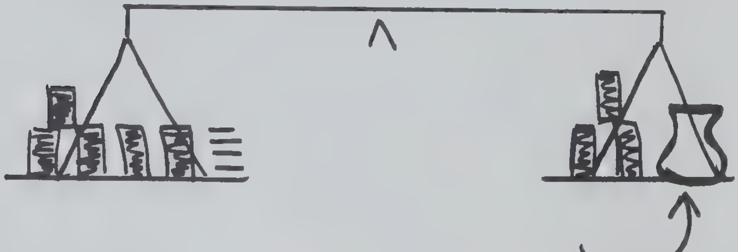
$$(5 \times \square) + 4 = (3 \times \square) + 16$$

Do you think their equation and picture are right?

Lex's equation

$(5\times\square)+4=(3\times\square)+16$

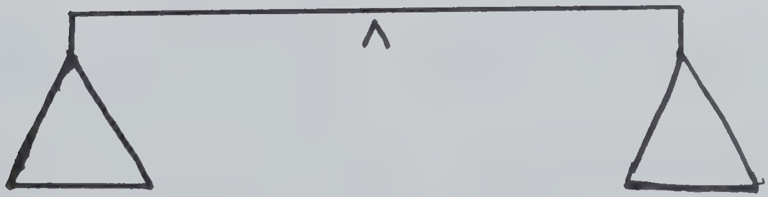
Debbie's balance picture



Bag with 16 loose washers

Jerry: You can take off 4 loose washers from each side.

49) What did Debbie's Balance Picture look like now?

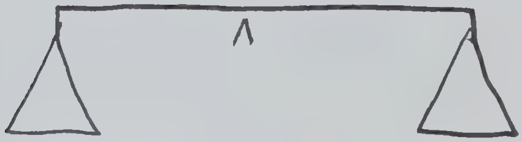


50) What did Lex's equation look like now?

Lex's equation

(5×□) ______________

Debbies Balance Picture



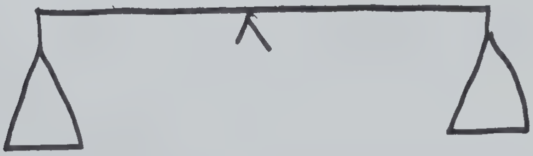
51) Marie: You can take off 3 rolls of washers from each side.

What balance picture did Debbie draw now? What equation did Lex write?

Lex's equation

______ = ______
? ?

Debbies Balance Picture



52) Can you find the truth set for the

equation $(2 \times \square) = 12$?

53) Hal says that Debbie and Lex have found the truth set for the wrong equation. Hal says they found the truth set for

$$2 \times \square = 12,$$

and they should have found the truth set for

$$(5 \times \square) + 4 = (3 \times \square) + 16.$$

What do you think?

54) What name do we give to two equations that have the same truth set?

55) What do we mean by a "transform operation"?

56) Paul says that adding 3 to both sides of an equation is one example of transform operation. Do you agree?

57) Can you give some more examples of transform operations?

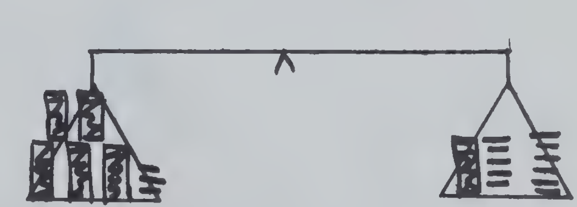
58) How many transform operations do you know?

59) John made these equations and pictures, in order to solve

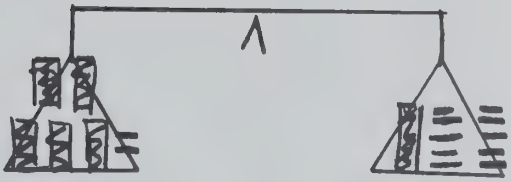
$$(5 \times \square) + 3 = (1 \times \square) + 11.$$

Do you agree with John's work?

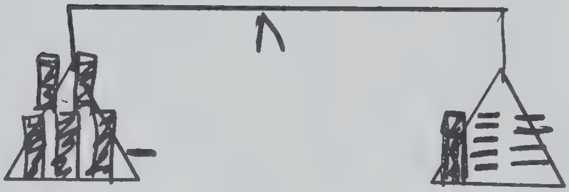
$(5 \times \square) + 3 = (1 \times \square) + 11$



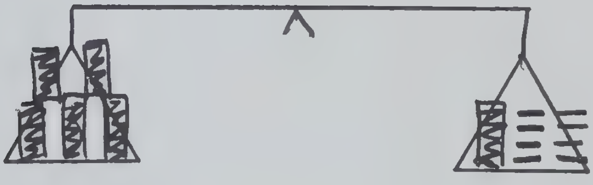
$(5 \times \square) + 2 = (1 \times \square) + 10$



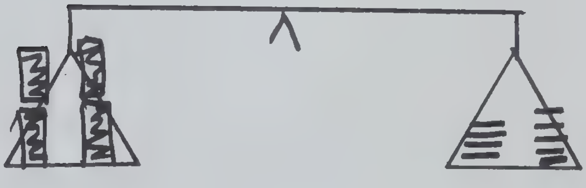
$(5 \times \square) + 1 = (1 \times \square) + 9$



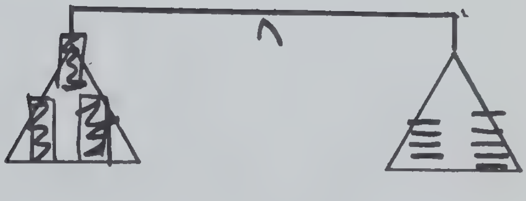
$(5 \times \square) = \square + 9$



$(4 \times \square) = (0 \times \square) + 9$



$(3 \times \square) = 9$



divide both sides by Three :

$\square = 3$

and so the truth set is

$$\{3\}.$$

40. Cindy's Machine

Can you find the truth set for these equations?

1) $(5\times\square)+3=(3\times\square)+15$

2) $(5\times\square)+8=(1\times\square)+16$

Can you make balance pictures for These equations?

3) $(5\times\square)+10=(3\times\square)+24$

4) $(5\times\square)+6=(1\times\square)+18$

5) What do we call two equations that both have the same truth set?

6) What do we mean by a "transform operation"?

7) Can you find the Truth set for

$2 \times \square = 8$?

8) Can you find the truth set for

$9 \times \square = 27$?

9) Suppose a and b are placeholders for some numbers. Can you find the truth set for

$a \times \square = b$?

10) Cindy says she has a kind of mathematical "machine" that will solve all equations of this type. Can you guess what Cindy's "machine" is?

11) Cindy says that her "machine" is this :

$$\square = \frac{b}{a}$$

Can you use Cindy's "machine" to solve

$$5 \times \square = 30 \text{ ?}$$

Can you use Cindy's machine to solve these equations:

12) $7 \times \square = 35$
13) $2 \times \square = 8$
14) $9 \times \square = 27$
15) $131 \times \square = 278$
16) $21 \times \square = 98$
17) Make up a problem yourself.

Can you solve it by Cindy's machine?

18) Can you make up a "machine" to solve

$$(a \times \square) + b = c?$$

19) Jerry says this machine will work:

$$\square = \frac{b+c}{a}$$

Does Jerry's machine really work?

20) Al says this machine will work:

$$\square = \frac{a}{b+c}.$$

Does Al's machine really work?

21) Can you draw a balance picture for $(5 \times \square) + 8 = 23$?

22) Can you use transform operations to simplify $(5 \times \square) + 8 = 23$?

At each step, make a balance picture.

23) Hal says that the transform operations that he used were:

"Subtract the same number from both sides. Divide both sides by the same number."

Are these the ones you used?

24) Can you make a "machine" to solve $(a \times \square) + b = c$?

25) Can you make a "machine" to solve this equation:

$$(a \times \square) + b = (c \times \square)\ ?$$

26) Fred made this machine:

$$\square = \frac{b+c}{a}.$$

Does Fred's machine really work?

27) Can you make a balance picture for

$$(5 \times \square) + 18 = (7 \times \square)?$$

28) Can you use transform operations to simplify $(5 \times \square) + 18 = (7 \times \square)$?
What transform operations did you use?

Can you make a "machine" to solve each of these equations?

29) $(a \times \square) + (b \times \square) = c$

30) $(a \times \square) + b = c + d$

31) $\square + a = b$

32) $a + b + (c \times \square) = d$

33) $(a \times \square) + (b \times \square) = c + d$

34) $a + b = (c \times \square) + d + e$

35) $(a \times \square) + b = (c \times \square) + d$

36) $\frac{a}{\square} = b$

37) $\frac{a}{\square} = b + c$

38) $(a \times \square) = b + c + d$

39) $(a \times \square) + (b \times \square) + (c \times \square) = d$

40) $(a \times \square) + (b \times \square) = c + (d \times \square)$

41) $(a \times \square) + (b \times \square) + c = (d \times \square)$

41. DERIVATIONS

1) What do we mean by an "identity"?

2) Can you write down twenty identities?

3) How many more identities do you suppose there are?

4) Lex has a special list of identities:

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

$$\square \times 1 = \square$$

$$\square + 0 = \square$$

$$\square \times 0 = 0$$

$$\frac{\square}{\square} = 1$$

$$\square = \square$$

$$\begin{aligned} 1+1 &= 2 \\ 2+1 &= 3 \\ 3+1 &= 4 \\ 4+1 &= 5 \\ \vdots \quad & \quad \vdots \end{aligned}$$

What do you suppose is special about Lex's list?

5) Marie says that most of Lex's identities are so important that they have names. Do you Know any of the names?

6) What do we mean by a "derivation"?

7) Why are "derivations" important?

Can you write a one step at a time derivation for each of these identities?

8) $A+(B\times C)=(C\times B)+A$

9) $(A+B)\times(A+B)=(A+A)+[(B+B)\times A]+(B\times B)$

10) $2\times 3=3+3$

11) $2\times 3=6$

12) $(A+B)\times(C+D)=(A\times C)+(A\times D)+(B\times C)+(B\times D)$

13) $(A+B)+(C+D)=(D+C)+(B+A)$

14) Can you guess how to finish the right-hand side of this identity?

$(A+B+C)\times(D+E+F)=(A\times D)+$ ____________

42. THE "ASSOCIATIVE" LAWS

1) Debbie says that Lex's list of identities is not really complete. She says it should also include this one:

$$\square + (\triangle + \triangledown) = (\square + \triangle) + \triangledown$$

What do you think?

2) Janet say that Debbie's identity has a name. Do you know what it is called?

3) Can you make up an "associative" law for multiplication?

Can you make up a derivation for each of these identities?

4) $(A+B)+(C+D)=(D+A)+(C+B)$

5) $(A+B)+(B+C)=(A+C)+(2\times B)$

6) $\square+\square=2\times\square$

7) $2\times 4=4+4$

8) $(3\times\square)+(2\times\square)=(5\times\square)$

9) $(a\times\square)+(b\times\square)=(a+b)\times\square$

43. AREA

1) Who is the biggest student in class?

2) John says it all depends. Do you mean the tallest, or the heaviest, or the oldest, or the strongest? What do you think?

3) Which is bigger, a pocket jack-knife, or a piece of paper?

4) Which is bigger, the state of Rhode Island, or the Mississippi River?

5) Which is bigger, an automobile or a telephone pole?

6) Do you know what we mean by length?

7) Do you know what we mean by area?

8) Do you know what we mean by volume?

9) Can you think of something that is very tall, but has a small volume?

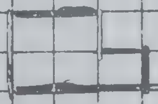
10) Can you think of something that has a big area, but has a small volume?

11) Can you think of something that has a smaller area, but a larger volume?

Now -- use a piece of graph paper:

12) Can you draw a figure that has an area of 5 squares?

13) Al drew this picture. Does it have an area of five squares?

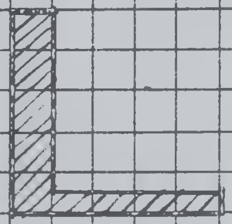


14) Did anyone in your class draw a different picture, instead of the one that Al drew? Did their picture have an area of five squares?

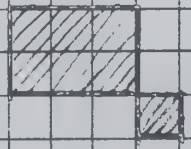
#44. HOW MANY SQUARES?

1) Can you draw a figure that has an area of seven squares?

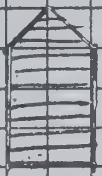
2) Debbie drew this picture. Does it have an area of seven squares?



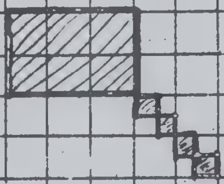
3) Marc drew this picture. Does it have an area of seven squares?



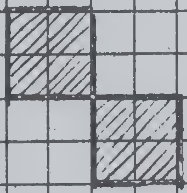
4) Ellen drew this picture. What area does it have?



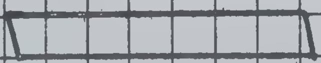
5) Jerry drew this picture. What area does it have?



6) Tony drew this picture. What area does it have?

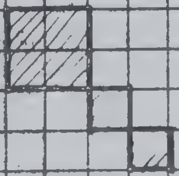


7) Alec drew this picture. What area does it have?



Can you find the areas of these figures?

8)



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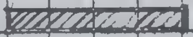

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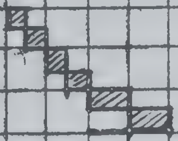

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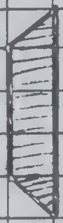

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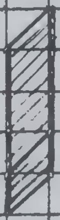

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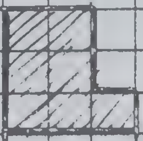

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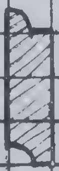

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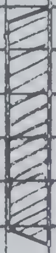

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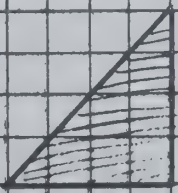

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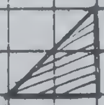

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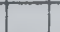
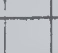
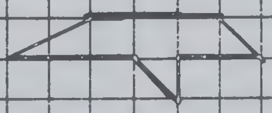


19)



20) Jerry says his picture has an area of five squares:



Do you agree?

21) Can you draw a figure with an area of four squares?

22) Can you draw a figure with an area of 2½ squares?

23) Do you know what a "triangle" is?

24) Can you draw a triangle that has an area of 2 squares?

25) Do you know what a "rectangle" is?

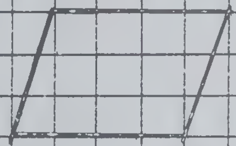
26) Do you know why we use the name "rectangle"? What is a "right angle"? Can you draw a

right angle?

27) Can you draw a rectangle that has an area of 8 squares?

28) Do you Know what a "parallelogram" is?

29) Jerry says this is a parallelogram. Do you agree?



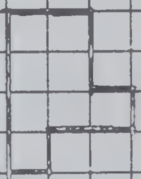
30) Can you draw a parallelogram that has an area of 5 squares?

31) Do you Know what a "trapezoid" is?

32) Hal says that a trapezoid has two parallel sides. Is this a good description of a trapezoid?

33) Can you draw a trapezoid that has an area of 8 squares?

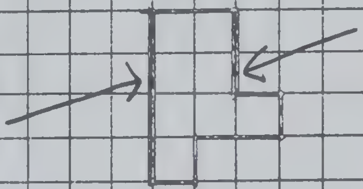
34) Aaron drew this picture:



Is this a trapezoid with an area of 8 squares?

35) Jerry says that Aaron's figure is not a trapezoid. What do you think?

36) Aaron says his figure has these two parallel sides, so it must be a trapezoid.



37) Hal says that what he meant to say was that a trapezoid is a four-sided figure with two parallel sides. What do you think?

38) Is a rectangle also a trapezoid?

39) Is a parallelogram also a trapezoid?

40) Is every trapezoid also a rectangle?

41) Is every trapezoid also a parallelogram?

42) Is every square also a rectangle?

43) Is every square also a parallelogram?

44) Can you draw a trapezoid with an area of six squares?

45) Can you draw a trapezoid that is also a parallelogram?

46) Can you draw a trapezoid that is not a parallelogram?

47) Can you draw a triangle that has an area of 3½ squares?

48) Can you draw a figure that has an area

of 1/4 square?

49) Can you draw a rectangle that has an area of 60 squares?

50) Can you draw a triangle that has an area of 30 squares?

45. SIMULTANEOUS EQUATIONS

Can you make a graph for the truth set of each equation?

1) $\triangle = ({}^{+}3 \times \square) + {}^{+}2$

2) $\triangle = ({}^{+}5 \times \square) + {}^{-}3$

3) $\triangle = ({}^{-}2 \times \square) + {}^{+}4$

4) Lex says that he Knows that $(\boxed{{}^{+}2}, \overset{{}^{+}8}{\triangle})$ is a point on the line for question number 1. Therefore, he says he Knows something else about $(\boxed{{}^{+}2}, \overset{{}^{+}8}{\triangle})$. What else does Lex Know?

5) Mary says that if you substitute $(\boxed{{}^{+}7}, \overset{{}^{+}2}{\triangle})$ into the equation of question 2, you will get a true statement. Does the

point $\left(\boxed{{}^{+}1}, \triangle\,{}^{+}2\right)$ lie on the graph for question 2?

6) Can you find one pair of numbers to fill $\square, \triangle$ so that both of these statements will be true?

$$\triangle = ({}^{+}3 \times \square) + {}^{-}2$$

$$\triangle = ({}^{-}2 \times \square) + {}^{+}8$$

7) Can you find one pair of numbers that will make both statements true?

$$\triangle = ({}^{+}2 \times \square) + {}^{+}1$$

$$\triangle = ({}^{-}2 \times \square) + {}^{+}5$$

8) Can you find one pair of numbers that will make both statements true?

$$\triangle = ({}^{+}3 \times \square) + {}^{+}1$$

$$\triangle = ({}^{-}1 \times \square) + {}^{+}9$$

9) Can you find one pair of numbers that will make both statements true?

$$\triangle = ({}^{+}4 \times \square) + {}^{+}5$$

$$\triangle = ({}^{-}2 \times \square) + {}^{+}5$$

10) Have you found a "trick method" for solving these equations? Can you describe it?

11) Can you solve this pair:

$$\triangle = ({}^{+}2 \times \square) + {}^{+}1$$

$$\triangle = ({}^{+}4 \times \square) + {}^{+}4$$

12) Can you solve this pair?

$$\triangle = ({}^{+}2 \times \square) + {}^{+}7$$

$$\triangle = ({}^{+}4 \times \square) + {}^{+}12$$

13) Can you solve this pair:

$$\triangle = \square$$

$$\triangle = ({}^{+}3 \times \square) + {}^{-}4$$

14) Can you solve this pair:

$$\triangle = \square$$

$$\triangle = ({}^{+}4 \times \square) + {}^{+}3$$

15) Can you solve this pair:

$$\triangle = \square + 1$$

$$\triangle = ({}^{+}3 \times \square) + {}^{-}3$$

16) Can you solve this pair:

$$\triangle = 2 \times \square$$

$$\triangle = (4 \times \square) + 7$$

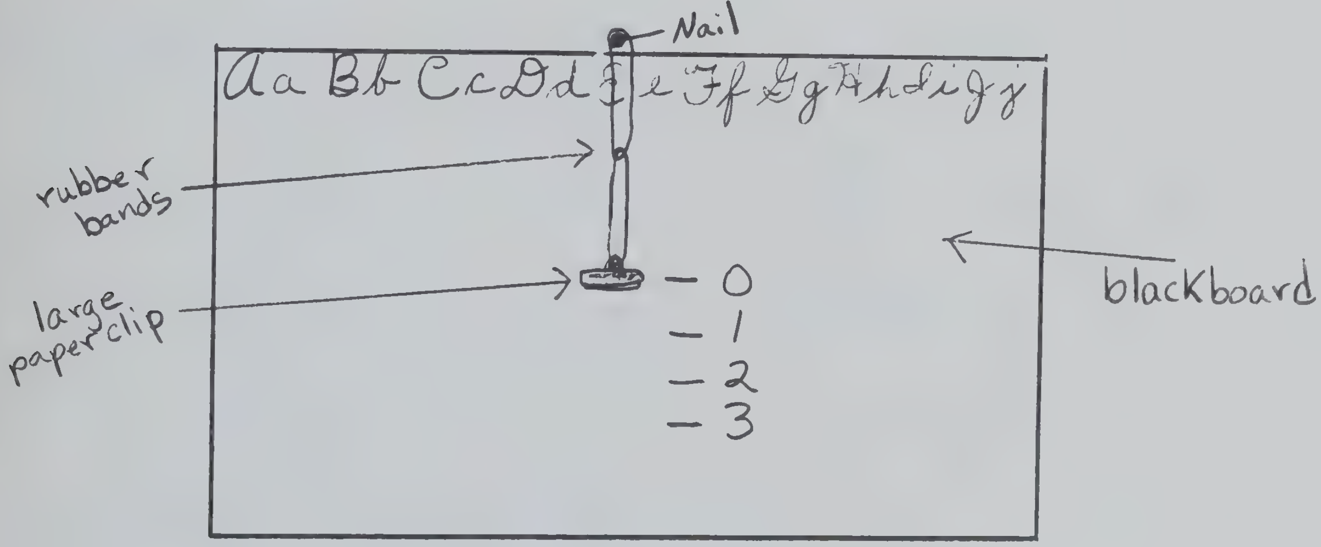
17) Can you solve this pair:

$$\square + \triangle = 5$$

$$\square - \triangle = 3$$

46. A LAW OF PHYSICS

1) Jerry wants to weigh some bags of washers. He is trying to make a balance like this:



How can he decide where to put the marks "0", "1", "2", "3", and so on?

2) Tony says that these marks should be equally spaced. Do you agree?

3) Can you fill in this table?

□ is a placeholder for the number of washers.

□	△
0	0
1	
2	
3	
4	
5	
6	
7	

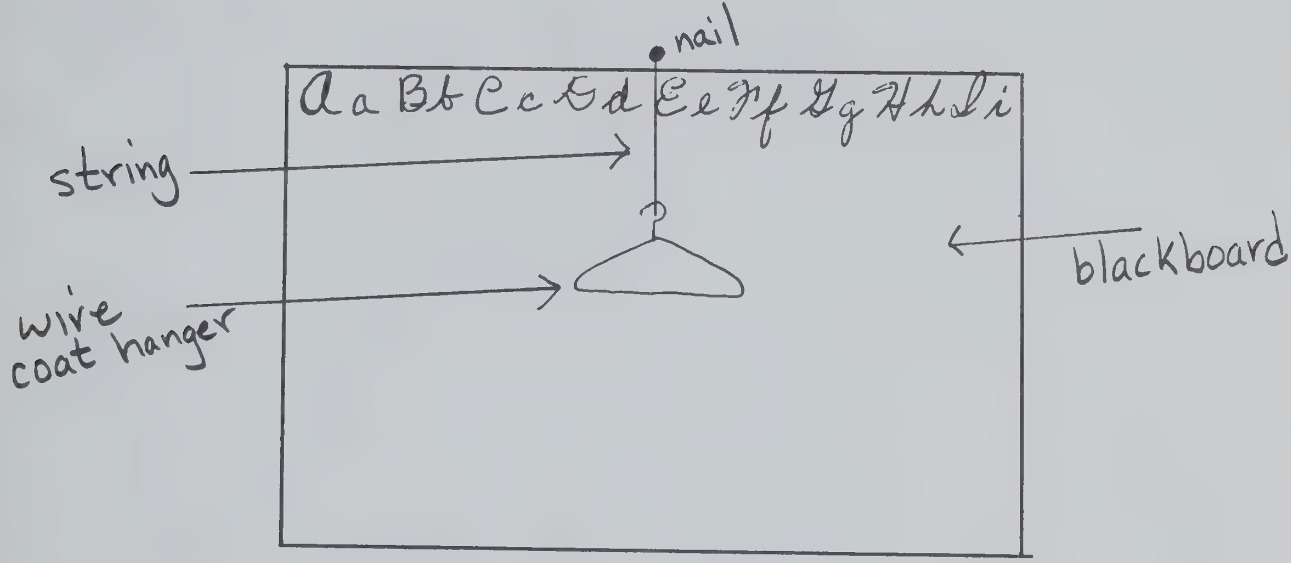
△ is a placeholder for the distance that the rubber bands stretch.

4) Can you represent these numbers on a graph?

5) Ellen says that a famous physicist named Hooke described how things stretch. Do you know what Hooke said?

6) Sarah wants to make a balance also,

but she wants to use two wire coat hangers instead of rubber bands:



Can you finish Sarah's balance?

7) Can you make a table for the number of washers (□) and the distance that the clip moves (△), for Sarah's balance?

8) Can you represent these numbers on a graph?

9) Can you write the equation for this?

10) Should the numbers "0", "1", "2", ... on Sarah's balance be equally spaced?

11) Can you state Hooke's law?

Q. 2.7

(c) Sketch the [illegible]

the Spacings be equally spaced?

What you state Hooke's law?

DATE DUE

MAY 14 '63
OV 16 '63
MAR 16 '64
APR 6 '64
APR 22 '64
AY 11 '64
JUL 13 '64
OCT 22 '64
MAR 5 '65
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